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Smoothness of generalized solutions for higher-order elliptic equations with nonlocal boundary conditions ☆

Pavel Gurevich

Interdisciplinary Center for Scientific Computing, University of Heidelberg, Im Neuenheimer Feld 368, D-69120 Heidelberg, Germany

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Abstract

The smoothness of generalized solutions for higher-order elliptic equations with nonlocal boundary conditions is studied in plane domains. Necessary and sufficient conditions upon the right-hand side of the problem and nonlocal operators under which the generalized solutions possess an appropriate smoothness are established.

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1. Introduction

In 1932, Carleman [7] considered the problem of finding a harmonic function, in a plane bounded domain, satisfying a nonlocal condition which connects the values of the unknown function at different points of the boundary. Further investigation of elliptic problems with transformations mapping a boundary onto itself as well as with abstract nonlocal conditions has been carried out by Vishik [34], Browder [6], Beals [3], Antonevich [2], and others.

In 1969, Bitsadze and Samarskii [5] considered the following nonlocal problem arising in the plasma theory: to find a function $u(y_1, y_2)$ harmonic on the rectangular $G = \{y \in \mathbb{R}^2\colon\ -1 < y_1 < 1,\ 0 < y_2 < 1\}$, continuous on $\overline{G}$, and satisfying the relations

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E-mail address: pavel.gurevich@iwr.uni-heidelberg.de.

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$$u(y_1,0)=f_1(y_1),\qquad u(y_1,1)=f_2(y_1),\quad -1<y_1<1,$$
$$u(-1,y_2)=f_3(y_2),\qquad u(1,y_2)=u(0,y_2),\quad 0<y_2<1,$$

where f_1, f_2, f_3 are given continuous functions. This problem was solved in [5] by reducing it to a Fredholm integral equation and using the maximum principle. For arbitrary domains and general nonlocal conditions, such a problem was formulated as an unsolved one (see also [8,23]). Different generalizations of nonlocal problems with transformations mapping the boundary inside the closure of a domain were studied by many authors [9,17,18,22].

The most complete theory for elliptic equations of order $2m$ with general nonlocal conditions was developed by Skubachevskii and his students [14,20,25–30]: a classification with respect to types of nonlocal conditions was suggested, the Fredholm solvability in the corresponding spaces was investigated, and asymptotics of solutions near special conjugation points was obtained.

Note that, besides the plasma theory, nonlocal elliptic problems have interesting applications to biophysics and theory of diffusion processes [10,11,24,32,33], control theory [1,4], theory of functional differential equations, mechanics [30], and so on.

The most difficult situation in the theory of nonlocal problems is that where the support of nonlocal terms can intersect the boundary of a domain. In this case, solutions of nonlocal problems can have power-law singularities near some points of the boundary even if the right-hand side is infinitely differentiable and the boundary is infinitely smooth [16,26,31]. This gives rise to the question of distinguishing some classes of nonlocal problems whose solutions are sufficiently smooth, provided that the right-hand side of the problem is smooth. Until now, this issue was studied only for nonlocal perturbations of the Dirichlet problem for second-order elliptic equations [16,31].

In the present paper, we investigate the smoothness of solutions for *elliptic equations of higher order with general nonlocal conditions in plane domains.* Unlike the theory of elliptic problems in nonsmooth domains, the violation of smoothness of solutions for nonlocal problems is connected not only with the fact that the boundary may contain singular points but rather with the presence of nonlocal terms in the boundary conditions.

We illustrate some of the occurring phenomena with the following example. Let $\partial G=\Gamma_1\cup\Gamma_2\cup\{g,h\}$, where Γ_i are open (in the topology of ∂G) C^∞ curves; g,h are the end points of the curves $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$. Suppose that the domain G is the plane angle of opening π in some neighborhood of each of the points g and h. We deliberately take a smooth domain to illustrate how the nonlocal terms can affect the smoothness of solutions. Consider the following problem in the domain G:

$$\Delta u=f_0(y)\quad (y\in G), \tag{1.1}$$

$$u|_{\Gamma_1}+b_1(y)u\big(\Omega_1(y)\big)\big|_{\Gamma_1}+a(y)u\big(\Omega(y)\big)\big|_{\Gamma_1}=f_1(y)\quad (y\in\Gamma_1),$$
$$u|_{\Gamma_2}+b_2(y)u\big(\Omega_2(y)\big)\big|_{\Gamma_2}=f_2(y)\quad (y\in\Gamma_2). \tag{1.2}$$

Here b_1, b_2, and a are real-valued C^∞ functions; Ω_i (Ω) are C^∞ diffeomorphisms taking some neighborhood $\mathcal{O}_i$ ($\mathcal{O}_1$) of the curve Γ_i (Γ_1) onto the set $\Omega_i(\mathcal{O}_i)$ ($\Omega(\mathcal{O}_1)$) in such a way that $\Omega_i(\Gamma_i)\subset G$, $\Omega_i(g)=g$, $\Omega_i(h)=h$, and the transformation Ω_i, near the points g,h, is the rotation of the boundary Γ_i through the angle $\pi/2$ inwards the domain G (respectively, $\Omega(\Gamma_1)\subset G$, $\overline{\Omega(\Gamma_1)}\cap\{g,h\}=\emptyset$, and the approach of the curve $\Omega(\overline{\Gamma_1})$ to the boundary ∂G can be arbitrary, cf. [26,28]), see Fig. 1.

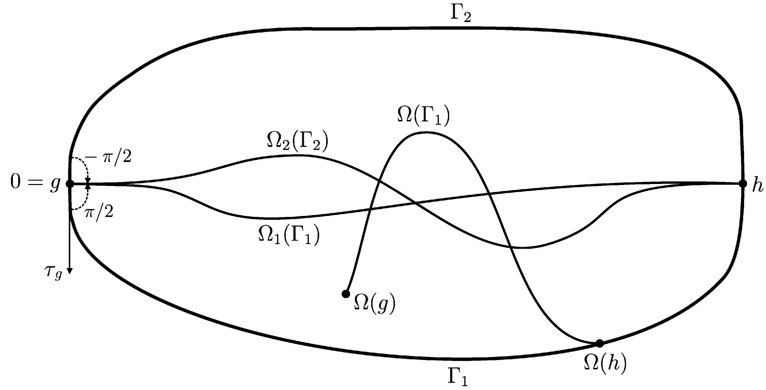

Fig. 1. Domain G with boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \{g, h\}$.

We say that g and h are the *points of conjugation of nonlocal conditions* because they divide the curves on which different nonlocal conditions are set. The closure of the set

$$\bigcup_{i=1,2} \{y \in \Omega_i(\Gamma_i)\colon\ b_i(\Omega_i^{-1}(y)) \neq 0\} \cup \{y \in \Omega(\Gamma_1)\colon\ a(\Omega^{-1}(y)) \neq 0\}$$

is referred to as the *support of nonlocal terms*.

Denote by $W^k(G) = W_2^k(G)$ the Sobolev space. We say that a function $u \in W^1(G)$ is a *generalized solution* of problem (1.1), (1.2) with right-hand side $f_0 \in L_2(G)$, $f_i \in W^{1/2}(\Gamma_i)$ if u satisfies nonlocal conditions (1.2) (the equalities are understood as those in $W^{1/2}(\Gamma_i)$) and Eq. (1.1) in the sense of distributions. Assume that $f_i \in W^{3/2}(\Gamma_i)$. Then one can show that any generalized solution of problem (1.1), (1.2) belongs to the space W^2 outside of an arbitrarily small neighborhood of the points g and h. Clearly, the behavior of solutions near the points g and h is affected by the behavior of the coefficients b_1, b_2, and a near these points. However, the influence of the coefficients b_i is principally different from that of the coefficient a. This phenomenon is explained by the fact that the coefficients b_i (for y being in a small neighborhood of the points g and h) correspond to nonlocal terms supported *near* the set $\{g, h\}$ (in the general case, such terms correspond to operators $\mathbf{B}_{i\mu}^1$), whereas the coefficient a corresponds to a nonlocal term supported *outside* of some neighborhood of the set $\{g, h\}$ (in the general case, such terms correspond to abstract operators $\mathbf{B}_{i\mu}^2$).

It was proved in [16] that the smoothness of generalized solutions preserves if $b_1(g)+b_2(g) \leqslant -2$ or $b_1(g) + b_2(g) > 0$ and can be violated if $-2 < b_1(g) + b_2(g) < 0$. If $b_1(g) + b_2(g) = 0$, we have the "border" case: the smoothness of generalized solutions depends on the fulfillment of some integral consistency condition imposed on the right-hand sides f_i and the coefficients b_i.

Now we illustrate another phenomenon arising in the border case. Assume that $b_1(y) \equiv b_2(y) \equiv 0$. Let $a(y) = 0$ in some neighborhood of the point h and $\Omega(g) \in G$. Then the *support of nonlocal terms lies strictly inside the domain G*. However, if $a(g) \neq 0$ or $(\partial a/\partial \tau_g)|_{y=g} \neq 0$, where τ_g denotes the unit vector tangent to ∂G at the point g, then the smoothness of generalized solutions of problem (1.1), (1.2) (even with homogeneous nonlocal conditions: $\{f_i\} = 0$) can be violated.

The phenomena similar to the above occur in the case of elliptic equations of order $2m$ with general nonlocal conditions, which we study in the present paper. In Section 2, we provide the setting of nonlocal problem and introduce the notion of a generalized solution $u \in W^\ell(G)$ of the problem for any integral $0 \leqslant \ell \leqslant 2m-1$.

It turns out that the smoothness of generalized solutions essentially depends on the location of eigenvalues and the structure of root functions of some auxiliary nonlocal operator $\tilde{\mathcal{L}}(\lambda)$, $\lambda \in \mathbb{C}$, corresponding to the conjugation points.

Let Λ denote the set of all eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ lying in the strip $1-2m < \operatorname{Im}\lambda < 1-\ell$ (this set might be empty). In Section 3 we assume that the line $\operatorname{Im}\lambda = 1-2m$ has no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ and find sufficient conditions on the eigenvalues from the set Λ under which any generalized solution of nonlocal problem belongs to $W^{2m}(G)$.

In Section 4, we investigate the "border" case in which the line $\operatorname{Im}\lambda = 1-2m$ contains the unique eigenvalue $i(1-2m)$ of $\tilde{\mathcal{L}}(\lambda)$ and this eigenvalue is proper (see Definition 3.1). We show that, under the same conditions on the eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ as in Section 3, the smoothness of generalized solutions preserves if and only if the right-hand side of the problem and the coefficients at the nonlocal terms satisfy some integral consistency conditions near the conjugation points.

In Section 5, we show that the sufficient conditions from the previous sections are also necessary for any generalized solution to be smooth.

Some facts concerning the functional spaces and model nonlocal problems in plane angles which we use throughout the paper are collected in Appendix A.

2. Setting of nonlocal problems in bounded domains

2.1. Setting of the problem

Let X be a domain in $\mathbb{R}^n$, $n=1,2$. Denote by $C_0^\infty(X)$ the set of functions infinitely differentiable on $\overline{X}$ and compactly supported in X. If M is a union of finitely many points (for $n=1,2$) or curves (for $n=2$) lying in $\overline{X}$, we denote by $C_0^\infty(\overline{X}\setminus M)$ the set of functions infinitely differentiable on $\overline{X}$ and compactly supported in $\overline{X}\setminus M$.

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. Consider a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^N \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞ curves. Assume that the domain G is a plane angle in some neighborhood of each point $g \in \mathcal{K}$.

For an integral $k \geqslant 0$, denote by $W^k(G) = W_2^k(G)$ the Sobolev space with the norm

$$\|u\|_{W^k(G)} = \left(\sum_{|\alpha|\leqslant k} \int_G \left|D^\alpha u(y)\right|^2 dy\right)^{1/2}$$

(set $W^0(G) = L_2(G)$ for $k=0$), where $\alpha = (\alpha_1,\dots,\alpha_n)$, $|\alpha| = \alpha_1+\dots+\alpha_n$, $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$.

For an integral $k \geqslant 1$, we introduce the space $W^{k-1/2}(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{G}$ with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|u\|_{W^k(G)} \quad \bigl(u \in W^k(G)\colon u|_\Gamma = \psi\bigr).$$

Along with Sobolev spaces, we will use weighted spaces (the Kondrat'ev spaces). Let $Q = \{y \in \mathbb{R}^2\colon r>0,\ |\omega|<\omega_0\}$, $Q = \{y \in \mathbb{R}^2\colon 0<r<d,\ |\omega|<\omega_0\}$, $0<\omega_0<\pi$, $d>0$, or $Q=G$.

We denote by $\mathcal{M}$ the set $\{0\}$ in the first and second cases and the set $\mathcal{K}$ in the third case. Introduce the space $H_a^k(Q) = H_a^k(Q, \mathcal{M})$ as a completion of the set $C_0^\infty(\overline{Q} \setminus \mathcal{M})$ with respect to the norm

$$\|u\|_{H_a^k(Q)} = \left(\sum_{|\alpha| \leqslant k} \int_Q \rho^{2(a-k+|\alpha|)} \left| D^\alpha u(y) \right|^2 dy \right)^{1/2},$$

where $a \in \mathbb{R}$, $k \geqslant 0$ is an integral, and $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{M})$. For an integral $k \geqslant 1$, denote by $H_a^{k-1/2}(\Gamma)$ the set of traces on a smooth curve $\Gamma \subset \overline{Q}$ with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Gamma)} = \inf \|u\|_{H_a^k(Q)} \quad \bigl(u \in H_a^k(Q)\colon\ u|_\Gamma = \psi\bigr). \tag{2.1}$$

Denote by $\mathbf{P}(y, D_y) = \mathbf{P}(y, D_{y_1}, D_{y_2})$ and $B_{i\mu s}(y, D_y) = B_{i\mu s}(y, D_{y_1}, D_{y_2})$ differential operators of order $2m$ and $m_{i\mu}$ ($m_{i\mu} \leqslant 2m - 1$), respectively, with complex-valued C^∞ coefficients ($i = 1, \dots, N$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$). Here $D_y = (D_{y_1}, D_{y_2})$, $D_{y_j} = -i\partial/\partial y_j$.

We assume that the following condition holds for the operators $\mathbf{P}(y, D_y)$ and $B_{i\mu 0}(y, D_y)$ (these operators will correspond to the "local" elliptic problem).

Condition 2.1. *The operator $\mathbf{P}(y, D_y)$ is properly elliptic on $\overline{G}$, and the system $\{B_{i\mu 0}(y, D_y)\}_{\mu=1}^m$ satisfies the Lopatinsky condition with respect to the operator $\mathbf{P}(y, D_y)$ for all $i = 1, \dots, N$ and $y \in \overline{\Gamma_i}$.*

We denote

$$\mathbf{B}_{i\mu}^0 u = B_{i\mu 0}(y, D_y)u, \quad y \in \Gamma_i,\ i = 1, \dots, N,\ \mu = 1, \dots, m.$$

For any closed set $\mathcal{M}$, we denote its ε-neighborhood by $\mathcal{O}_\varepsilon(\mathcal{M})$, i.e.,

$$\mathcal{O}_\varepsilon(\mathcal{M}) = \bigl\{y \in \mathbb{R}^2\colon \operatorname{dist}(y, \mathcal{M}) < \varepsilon\bigr\}, \quad \varepsilon > 0.$$

Now we introduce operators corresponding to nonlocal terms supported near the set $\mathcal{K}$. Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be C^∞ diffeomorphisms taking some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and

$$\Omega_{is}(g) \in \mathcal{K} \quad \text{for } g \in \overline{\Gamma_i} \cap \mathcal{K}. \tag{2.2}$$

Thus, the transformations Ω_{is} take the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ strictly inside the domain G and the set of their end points $\overline{\Gamma_i} \cap \mathcal{K}$ to itself.

Let us specify the structure of the transformations Ω_{is} near the set $\mathcal{K}$. Denote by Ω_{is}^{+1} the transformation $\Omega_{is} : \mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$ and by $\Omega_{is}^{-1} : \Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ the inverse transformation. The set of points $\Omega_{i_q s_q}^{\pm 1}(\dots \Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ ($1 \leqslant s_j \leqslant S_{i_j}$, $j = 1, \dots, q$) is said to be an *orbit* of the point $g \in \mathcal{K}$ and denoted by $\operatorname{Orb}(g)$. In other words, the orbit $\operatorname{Orb}(g)$ is formed by the points (of the set $\mathcal{K}$) that can be obtained by consecutively applying the transformations $\Omega_{i_j s_j}^{\pm 1}$ to the point g.

It is clear that either $\operatorname{Orb}(g) = \operatorname{Orb}(g')$ or $\operatorname{Orb}(g) \cap \operatorname{Orb}(g') = \varnothing$ for any $g, g' \in \mathcal{K}$. In what follows, we assume that the set $\mathcal{K}$ consists of one orbit (the results are easy to generalize for the

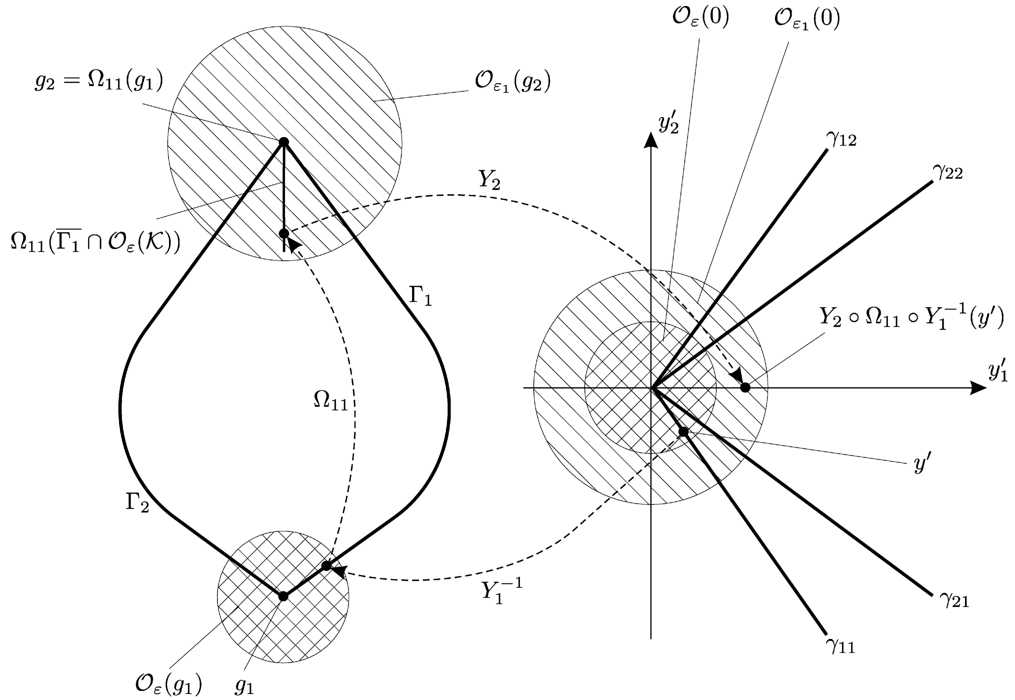

Fig. 2. The transformation $Y_2 \circ \Omega_{11} \circ Y_1^{-1} : \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$ is a composition of rotation and homothety.

case in which $\mathcal{K}$ consists of finitely many disjoint orbits, cf. Section 6 in [16]). To simplify the notation, we also assume that the set (orbit) $\mathcal{K}$ consists of N points: $g_1, \dots, g_N$.

Take a sufficiently small number ε (cf. Remark 2.3 in [16]) such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$, $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$, satisfying the following conditions:

(1) The domain G is a plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$.
(2) $\overline{\mathcal{O}_{\varepsilon_1}(g_j)} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_k)} = \emptyset$ for any $g_j, g_k \in \mathcal{K}$, $k \neq j$.
(3) If $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, we fix a transformation $Y_j : y \mapsto y'(g_j)$ which is a composition of the shift by the vector $-\overrightarrow{Og_j}$ and the rotation through some angle so that

$$Y_j\bigl(\mathcal{O}_{\varepsilon_1}(g_j)\bigr) = \mathcal{O}_{\varepsilon_1}(0), \qquad Y_j\bigl(G \cap \mathcal{O}_{\varepsilon_1}(g_j)\bigr) = K_j \cap \mathcal{O}_{\varepsilon_1}(0),$$
$$Y_j\bigl(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)\bigr) = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0) \quad (\sigma = 1 \text{ or } 2),$$

where

$$K_j = \bigl\{y \in \mathbb{R}^2\colon\ r > 0,\ |\omega| < \omega_j\bigr\}, \qquad \gamma_{j\sigma} = \bigl\{y \in \mathbb{R}^2\colon\ r > 0,\ \omega = (-1)^\sigma \omega_j\bigr\}.$$

Here (ω, r) are the polar coordinates and $0 < \omega_j < \pi$.

Let the following condition hold (see Fig. 2).

Condition 2.2. *Let $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}$; then the transformation*

$$Y_k \circ \Omega_{is} \circ Y_j^{-1} : \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$$

is the composition of rotation and homothety.

Remark 2.1. Condition 2.2, together with the fact that $\Omega_{is}(\Gamma_i) \subset G$, implies that if $g \in \Omega_{is}(\overline{\Gamma_i} \cap \mathcal{K}) \cap \overline{\Gamma_j} \cap \mathcal{K} \neq \emptyset$, then the curves $\Omega_{is}(\overline{\Gamma_i} \cap \mathcal{O}_\varepsilon(\mathcal{K}))$ and $\overline{\Gamma_j}$ intersect at nonzero angle at the point g.

We choose a number ε_0, $0 < \varepsilon_0 \leqslant \varepsilon$ possessing the following property: if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\varepsilon_0}(g_k) \subset \Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$. Consider a function $\zeta \in C^\infty(\mathbb{R}^2)$ such that

$$\zeta(y) = 1 \quad \bigl(y \in \mathcal{O}_{\varepsilon_0/2}(\mathcal{K})\bigr), \qquad \zeta(y) = 0 \quad \bigl(y \notin \mathcal{O}_{\varepsilon_0}(\mathcal{K})\bigr).$$

Introduce the nonlocal operators $\mathbf{B}^1_{i\mu}$ by the formulas

$$\begin{aligned} &\mathbf{B}^1_{i\mu} u = \sum_{s=1}^{S_i} \bigl(B_{i\mu s}(y, D_y)(\zeta u)\bigr)\bigl(\Omega_{is}(y)\bigr), && y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \\ &\mathbf{B}^1_{i\mu} u = 0, && y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}), \end{aligned}$$

where $(B_{i\mu s}(y, D_y)u)(\Omega_{is}(y)) = B_{i\mu s}(x, D_x)u(x)|_{x=\Omega_{is}(y)}$. Since $\mathbf{B}^1_{i\mu} u = 0$ for $\operatorname{supp} u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, we say that the operators $\mathbf{B}^1_{i\mu}$ *correspond to nonlocal terms supported near the set* $\mathcal{K}$.

Set $G_\rho = \{y \in G\colon \operatorname{dist}(y, \partial G) > \rho\}$ for $\rho > 0$. Consider operators $\mathbf{B}^2_{i\mu}$ satisfying the following condition (cf. [14,26,29]).

Condition 2.3. *There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that*

$$\bigl\|\mathbf{B}^2_{i\mu} u\bigr\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leqslant c_1 \|u\|_{W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})} \quad \forall u \in W^{2m}\bigl(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}\bigr), \tag{2.3}$$

$$\bigl\|\mathbf{B}^2_{i\mu} u\bigr\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \leqslant c_2 \|u\|_{W^{2m}(G_\rho)} \quad \forall u \in W^{2m}(G_\rho), \tag{2.4}$$

where $i = 1, \dots, N$, $\mu = 1, \dots, m$, and $c_1, c_2 > 0$ do not depend on u.

It follows from (2.3) that $\mathbf{B}^2_{i\mu} u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}^2_{i\mu}$ *correspond to nonlocal terms supported outside the set* $\mathcal{K}$.

We assume that Conditions 2.1–2.3 are fulfilled throughout.

We study the following nonlocal elliptic boundary-value problem:

$$\mathbf{P}(y, D_y) u = f_0(y) \quad (y \in G), \tag{2.5}$$

$$\mathbf{B}^0_{i\mu} u + \mathbf{B}^1_{i\mu} u + \mathbf{B}^2_{i\mu} u = f_{i\mu}(y) \quad (y \in \Gamma_i;\ i = 1, \dots, N;\ \mu = 1, \dots, m). \tag{2.6}$$

Note that the points g_j divide the curves on which different nonlocal conditions are set; therefore, it is natural to say that g_j, $j=1,\dots,N$, are the *points of conjugation of nonlocal conditions*.

Introduce the spaces of vector-valued functions

$$\mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)=\prod_{i=1}^{N}\prod_{\mu=1}^{m}W^{2m-m_{i\mu}-1/2}(\Gamma_i),$$

$$\mathcal{H}_a^{2m-\mathbf{m}-1/2}(\partial G)=\prod_{i=1}^{N}\prod_{\mu=1}^{m}H_a^{2m-m_{i\mu}-1/2}(\Gamma_i).$$

We will always assume that $\{f_0,f_{i\mu}\}\in L_2(G)\times\mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$.

From now on, we fix an integral number ℓ such that $0\leqslant\ell\leqslant 2m-1$.

Definition 2.1. A function u is called a *generalized solution* of problem (2.5), (2.6) with right-hand side $\{f_0,f_{i\mu}\}\in L_2(G)\times\mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$ if

$$u\in W^{\ell}(G)\cap W^{2m}\bigl(G\setminus\overline{\mathcal{O}_\delta(\mathcal{K})}\bigr)\quad\forall\delta>0 \tag{2.7}$$

and u satisfies relations (2.5) a.e. and equalities (2.6) in $W^{2m-m_{i\mu}-1/2}(\Gamma_i\setminus\overline{\mathcal{O}_\delta(\mathcal{K})})$ for all $\delta>0$.

Note that if u satisfies (2.7), then $\mathbf{B}^2_{i\mu}u\in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ due to (2.3) and $\mathbf{B}^1_{i\mu}u\in W^{2m-m_{i\mu}-1/2}(\Gamma_i\setminus\overline{\mathcal{O}_\delta(\mathcal{K})})$ for all $\delta>0$. Therefore, Definition 2.1 does make sense.

Remark 2.2. Let $W^{-k}(G)$, $k\geqslant 1$, denote the space adjoint to $W^k(G)$ with respect to the extension of the inner product in $L_2(G)$.

Denote by $H_a^{-(k-1/2)}(\Gamma_i)$, $k\geqslant 1$, the space adjoint to $H_{-a}^{k-1/2}(\Gamma_i)$ with respect to the extension of the inner product in $L_2(\Gamma_i)$.

One can show that $C^\infty(\overline{\Gamma_i})\subset H_\ell^{\ell-k+1/2}$, $k=1,\dots,2m$. Therefore, the norm

$$\|\mathbf{u}\|_{\mathbf{W}^\ell(G)}=\left(\|\mathbf{u}\|^2_{\mathbf{W}^\ell(G)}+\sum_{i=1}^{N}\sum_{k=1}^{2m}\bigl\|D_{\nu_i}^{k-1}\mathbf{u}\bigr\|^2_{H_\ell^{\ell-k+1/2}(\Gamma_i)}\right)^{1/2} \tag{2.8}$$

is finite for any $\mathbf{u}\in C^\infty(\overline{G})$, where ν_i is the outward normal to the piece Γ_i of the boundary and $D_{\nu_i}^{k-1}\mathbf{u}=(-i)^{k-1}\frac{\partial^{k-1}\mathbf{u}}{\partial\nu_i^{k-1}}|_{\Gamma_i}$. Denote by $\mathbf{W}^\ell(G)$ the completion of $C^\infty(\overline{G})$ in the norm (2.8).

It follows from (2.8) that the closure $\mathbf{S}$ of the mapping

$$\mathbf{u}\mapsto\bigl\{\mathbf{u}|_G,D_{\nu_i}^{k-1}\mathbf{u}\bigr\}\quad\bigl(\mathbf{u}\in C^\infty(\overline{G})\bigr)$$

establishes an isometric correspondence between $\mathbf{W}^\ell(G)$ and a subspace of the direct product

$$W^\ell(G)\times\prod_{i=1}^{N}\prod_{k=1}^{2m}H_\ell^{\ell-k+1/2}(\Gamma_i).$$

We will identify $\mathbf{u} \in \mathbf{W}^\ell(G)$ with $\mathbf{Su}$ and write $\mathbf{u} = \{u, u_{ik}\} \in \mathbf{W}^\ell(G)$.

Then, similarly to [21], one can introduce the concept of a strong generalized solution $\mathbf{u} \in \mathbf{W}^\ell(G)$ of problem (2.5), (2.6). Moreover, one can prove that if $\mathbf{u}$ is a strong generalized solution, then the component $u \in W^\ell(G)$ of the vector $\mathbf{u}$ is a generalized solution in the sense of Definition 2.1. Conversely, if $u \in W^\ell(G)$ is a generalized solution in the sense of Definition 2.1, then $\mathbf{u} = \{u, D_{\nu_i}^{k-1}u\}$ belongs to $\mathbf{W}^\ell(G)$ and is a strong generalized solution. Furthermore, if the function $\mathbf{v} = \{u, v_{ik}\} \in \mathbf{W}^\ell(G)$ (with the same first component u) is a strong generalized solution, then $\mathbf{u} = \mathbf{v}$, i.e., a generalized solution uniquely determines a strong generalized solution.

2.2. *Model problems*

When studying problem (2.5), (2.6), particular attention must be paid to the behavior of solutions near the set $\mathcal{K}$ of conjugation points. In this subsection, we consider corresponding model problems.

Denote by $u_j(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma_i}$, $y \in \mathcal{O}_\varepsilon(g_j)$, and $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then we denote the function $u(\Omega_{is}(y))$ by $u_k(\Omega_{is}(y))$. In this notation, nonlocal problem (2.5), (2.6) acquires the following form in the ε-neighborhood of the set (orbit) $\mathcal{K}$:

$$\mathbf{P}(y, D_y)u_j = f_0(y) \quad \big(y \in \mathcal{O}_\varepsilon(g_j) \cap G\big),$$

$$B_{i\mu 0}(y, D_y)u_j(y)|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} + \sum_{s=1}^{S_i}\big(B_{i\mu s}(y, D_y)(\zeta u_k)\big)\big(\Omega_{is}(y)\big)\big|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} = \psi_{i\mu}(y)$$

$$\big(y \in \mathcal{O}_\varepsilon(g_j) \cap \Gamma_i;\ i \in \{1 \leqslant i \leqslant N\colon\ g_j \in \overline{\Gamma_i}\};\ j = 1, \dots, N;\ \mu = 1, \dots, m\big),$$

where

$$\psi_{i\mu} = f_{i\mu} - \mathbf{B}^2_{i\mu}u.$$

Let $y \mapsto y'(g_j)$ be the change of variables described in Section 2.1. Set

$$K_j^\varepsilon = K_j \cap \mathcal{O}_\varepsilon(0), \qquad \gamma_{j\sigma}^\varepsilon = \gamma_{j\sigma} \cap \mathcal{O}_\varepsilon(0)$$

and introduce the functions

$$\begin{gathered} U_j(y') = u\big(y(y')\big), \qquad F_j(y') = f_0\big(y(y')\big), \quad y' \in K_j^\varepsilon, \\ F_{j\sigma\mu}(y') = f_{i\mu}\big(y(y')\big), \qquad B^u_{j\sigma\mu}(y') = \big(\mathbf{B}^2_{i\mu}u\big)\big(y(y')\big), \quad y' \in \gamma^\varepsilon_{j\sigma}, \\ \Psi_{j\sigma\mu}(y') = F_{j\sigma\mu}(y') - B^u_{j\sigma\mu}(y'), \quad y' \in \gamma^\varepsilon_{j\sigma}, \end{gathered} \tag{2.9}$$

where $\sigma = 1$ ($\sigma = 2$) if the transformation $y \mapsto y'(g_j)$ takes Γ_i to the side γ_{j1} (γ_{j2}) of the angle K_j. Denote y' by y again. Then, by virtue of Condition 2.2, problem (2.5), (2.6) acquires the form

$$\mathbf{P}_j(y, D_y)U_j = F_j(y) \quad \big(y \in K_j^\varepsilon\big), \tag{2.10}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U \equiv \sum_{k,s}\big(B_{j\sigma\mu ks}(y, D_y)U_k\big)(\mathcal{G}_{j\sigma ks}y) = \Psi_{j\sigma\mu}(y) \quad \big(y \in \gamma^\varepsilon_{j\sigma}\big). \tag{2.11}$$

Here (and below unless otherwise stated) $j,k = 1,\dots,N$; $\sigma = 1,2$; $\mu = 1,\dots,m$; $s = 0,\dots,S_{j\sigma k}$; $\mathbf{P}_j(y, D_y)$ and $B_{j\sigma\mu ks}(y, D_y)$ are differential operators of order $2m$ and $m_{j\sigma\mu}$ ($m_{j\sigma\mu} \leqslant 2m-1$), respectively, with C^∞ complex-valued coefficients; $\mathcal{G}_{j\sigma ks}$ is the operator of rotation by an angle $\omega_{j\sigma ks}$ and homothety with a coefficient $\chi_{j\sigma ks}$ ($\chi_{j\sigma ks} > 0$) in the y-plane. Moreover,

$$\left|(-1)^\sigma b_j + \omega_{j\sigma ks}\right| < b_k \quad \text{for } (k,s) \ne (j,0)$$

(cf. Remark 2.1) and

$$\omega_{j\sigma j0} = 0, \qquad \chi_{j\sigma j0} = 1$$

(i.e., $\mathcal{G}_{j\sigma j0} y \equiv y$).

Along with the operators $\mathbf{P}_j(y, D_y)$ and $\mathbf{B}_{j\sigma\mu}(y, D_y)$, we consider the operators

$$\mathcal{P}_j(D_y), \qquad \mathcal{B}_{j\sigma\mu}(D_y)U \equiv \sum_{k,s}\bigl(B_{j\sigma\mu ks}(D_y)U_k\bigr)(\mathcal{G}_{j\sigma ks} y), \tag{2.12}$$

where $\mathcal{P}_j(D_y)$ and $B_{j\sigma\mu ks}(D_y)$ are the principal homogeneous parts of the operators $\mathbf{P}_j(0, D_y)$ and $B_{j\sigma\mu ks}(0, D_y)$, respectively.

We write the operators $\mathcal{P}_j(D_y)$ and $B_{j\sigma\mu ks}(D_y)$ in the polar coordinates: $r^{-2m}\tilde{\mathcal{P}}_j(\omega, D_\omega, rD_r)$, $r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, rD_r)$, respectively, and consider the analytic operator-valued function[1]

$$\tilde{\mathcal{L}}(\lambda): \prod_{j=1}^{N} W^{2m}(-\omega_j, \omega_j) \to \prod_{j=1}^{N}\bigl(L_2(-\omega_j, \omega_j) \times \mathbb{C}^{2m}\bigr),$$

$$\tilde{\mathcal{L}}(\lambda)\varphi = \bigl\{\tilde{\mathcal{P}}_j(\omega, D_\omega, \lambda)\varphi_j, \tilde{\mathcal{B}}_{j\sigma\mu}(\omega, D_\omega, \lambda)\varphi\bigr\},$$

where $D_\omega = -i\partial/\partial\omega$, $D_r = -i\partial/\partial r$, and

$$\tilde{\mathcal{B}}_{j\sigma\mu}(\omega, D_\omega, \lambda)\varphi = \sum_{k,s}(\chi_{j\sigma ks})^{i\lambda - m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, \lambda)\varphi_k(\omega + \omega_{j\sigma ks})|_{\omega=(-1)^\sigma\omega_j}.$$

Spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ play a crucial role in the study of smoothness of generalized solutions. The following assertion is of particular importance (see Lemmas 2.1 and 2.2 in [27]).

Lemma 2.1. *For any $\lambda \in \mathbb{C}$, the operator $\tilde{\mathcal{L}}(\lambda)$ has the Fredholm property and* $\operatorname{ind}\tilde{\mathcal{L}}(\lambda) = 0$.

The set of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete. For any numbers $c_1 < c_2$, the band $c_1 < \operatorname{Im}\lambda < c_2$ contains at most finitely many eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

[1] Main definitions and facts concerning analytic operator-valued functions can be found in [12].

3. Preservation of smoothness of generalized solutions

3.1. Formulation of the main result

In this section, we study the case in which the following condition holds.

Condition 3.1. *The line* $\operatorname{Im}\lambda = 1-2m$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$.

Let $\lambda=\lambda_0$ be an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

Definition 3.1. (Cf. [14,19].) We say that λ_0 is a *proper eigenvalue* if none of the corresponding eigenvectors $\varphi(\omega)=(\varphi_1(\omega),\dots,\varphi_N(\omega))$ has an associated vector, while the functions $r^{i\lambda_0}\varphi_j(\omega)$, $j=1,\dots,N$, are homogeneous polynomials in y_1, y_2 (of degree $i\lambda_0\in\mathbb{N}\cup\{0\}$). An eigenvalue which is not proper is said to be *improper*.

Let Λ be the set of all eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ in the band $1-2m<\operatorname{Im}\lambda<1-\ell$ (this set can be empty). We also denote $i\Lambda=\{i\lambda\colon\ \lambda\in\Lambda\}$.

Condition 3.2. *All the eigenvalues from the set* Λ *are proper.*

In particular, Condition 3.2 implies that $\Lambda=\emptyset$ if $\ell=2m-1$ (e.g., if $\ell=m=1$, cf. [16]) and $i\Lambda\subset\{\ell,\dots,2m-2\}$ if $\ell\leqslant 2m-2$.

In the case where $\ell\leqslant 2m-2$, we will need some additional conditions.

Let $W^{-2m}(-\omega_j,\omega_j)$ be the space adjoint to $W^{2m}(-\omega_j,\omega_j)$. Consider the operator $(\tilde{\mathcal{L}}(\lambda))^*:\prod_{j=1}^N(L_2(-\omega_j,\omega_j)\times\mathbb{C}^{2m})\to\prod_{j=1}^N W^{-2m}(-\omega_j,\omega_j)$ which is adjoint to the operator $\tilde{\mathcal{L}}(\lambda)$.

For any $s\in\{\ell,\dots,2m-2\}$, we denote by J_s the set of all indices (j',σ',μ') such that

$$s\leqslant m_{j'\sigma'\mu'}-1. \tag{3.1}$$

We also denote by C_s the space of numerical vectors $\{c_{j\sigma\mu}\}$ with complex entries such that

$$c_{j'\sigma'\mu'}=0,\quad (j',\sigma',\mu')\in J_s.$$

Condition 3.3. *If* $\ell\leqslant 2m-2$, *then the following assertions hold for any* $s\in i\Lambda$:

(1) $J_s\neq\emptyset$.
(2) $\langle\{0,c_{j\sigma\mu}\},\psi\rangle=0$ *for all* $\{c_{j\sigma\mu}\}\in C_s$ *and* $\psi\in\ker(\tilde{\mathcal{L}}(-is))^*$.
(3) *Let* $\varphi_c\in\prod_j W^{2m}(-\omega_j,\omega_j)$ *denote a solution of the equation* $\tilde{\mathcal{L}}(-is)\varphi_c=\{0,c_{j\sigma\mu}\}$, *where* $\{c_{j\sigma\mu}\}\in C_s$ (*this solution exists due to item* (2) *and is defined up to an arbitrary element* $\varphi_0\in\ker\tilde{\mathcal{L}}(-is)$). *Then* $r^s\varphi_c(\omega)$ *is a homogeneous polynomial* (*of degree* s) *for any* $\{c_{j\sigma\mu}\}\in C_s$.

Remark 3.1. (1) Part (1) in Condition 3.3 is necessary for the fulfillment of part (2). This follows from Lemma 2.1.

(2) Part (2) is necessary and sufficient for the existence of solutions φ_c for all $\{c_{j\sigma\mu}\}$ in part (3).

Condition 3.4. *If $\ell \leqslant 2m-2$, then the following assertion holds for any $s \in \{\ell, \dots, 2m-2\} \setminus i\Lambda$. Let $\varphi_c \in \prod_j W^{2m}(-\omega_j, \omega_j)$ denote a solution*[2] *of the equation $\tilde{\mathcal{L}}(-is)\varphi_c = \{0, c_{j\sigma\mu}\}$, where $\{c_{j\sigma\mu}\} \in C_s$. Then $r^s\varphi_c(\omega)$ is a homogeneous polynomial (of degree s) for any $\{c_{j\sigma\mu}\} \in C_s$.*

Remark 3.2. Suppose that Condition 3.2 is fulfilled.

(1) If Conditions 3.3 and 3.4 hold, then the problem

$$\mathcal{P}_j(D_y)V = 0, \qquad \mathcal{B}_{j\sigma\mu}(D_y)V = c_{j\sigma\mu}r^{s-m_{j\sigma\mu}} \tag{3.2}$$

admits a solution $V(y)$ which is a homogeneous polynomial of degree s, provided that $\{c_{j\sigma\mu}\} \in C_s$, where $s = \ell, \dots, 2m-2$. Indeed, substituting a function $V = r^s\varphi_c(\omega)$ into (3.2), we obtain the equation $\tilde{\mathcal{L}}(-is)\varphi_s = \{0, c_{j\sigma\mu}\}$. Due to Conditions 3.3 and 3.4, this equation admits a solution φ_c such that the function $V = r^s\varphi_c(\omega)$ is a homogeneous polynomial of degree s.

(2) If Condition 3.3 or 3.4 fails, then there is a vector $\{c_{j\sigma\mu}\} \in C_s$ such that problem (3.2) admits a solution

$$V = r^s\varphi_c(\omega) + r^s(i\ln r)\sum_{n=1}^{J} c_n\varphi^{(n)}(\omega), \tag{3.3}$$

where $s \in \{\ell, \dots, 2m-2\}$, $c_n \in \mathbb{C}$, $\varphi_c, \varphi^{(n)} \in \prod_{j=1}^N W^{2m}(-\omega_j, \omega_j)$, and $J = J(s)$. Moreover, the function V is not a polynomial in y_1, y_2.

Indeed, if Condition 3.4 fails, then the assertion is evident (with $c_1 = \dots = c_J = 0$). Assume that Condition 3.3 fails. If parts (1) and (2) of Condition 3.4 hold while part (3) fails, then the assertion is evident again (with $c_1 = \dots = c_J = 0$). Let part (1) or (2) fail. In both cases, part (2) does not hold (see Remark 3.1). This means that there exists a proper eigenvalue $\lambda_s = -is \in \Lambda$ and a numerical vector $\{c_{j\sigma\mu}\} \in C_s$ such that $\{0, c_{j\sigma\mu}\}$ is not orthogonal to $\ker(\tilde{\mathcal{L}}(\lambda_s))^*$.

Let $\varphi^{(1)}, \dots, \varphi^{(J)}$ ($J \geqslant 1$) denote some basis in $\ker\tilde{\mathcal{L}}(\lambda_s)$. Since λ_s is a proper eigenvalue, none of the eigenvectors $\varphi^{(n)}$ has an associate vector. We substitute a function V given by (3.3) in Eqs. (3.2). Then we obtain

$$\tilde{\mathcal{L}}(\lambda_s)\varphi_c = \{0, c_{j\sigma\mu}\} - \sum_{n=1}^{J} c_n \frac{d\tilde{\mathcal{L}}(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_s}\varphi^{(n)}. \tag{3.4}$$

Note that $\dim\ker(\tilde{\mathcal{L}}(\lambda_s))^* = \dim\ker\tilde{\mathcal{L}}(\lambda_s) = J$ due to Lemma 2.1. Let $\psi^{(1)}, \dots, \psi^{(J)}$ denote a basis in $\ker(\tilde{\mathcal{L}}(\lambda_s))^*$. By Lemma 3.2 in [13], the matrix

$$\left\| \left\langle \frac{d\tilde{\mathcal{L}}(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_s}\varphi^{(n)}, \psi^{(k)} \right\rangle \right\|_{n,k=1,\dots,J}$$

is nondegenerate. Therefore, we can choose the constants c_n in such a way that the right-hand side in (3.4) is orthogonal to $\ker(\tilde{\mathcal{L}}(\lambda_s))^*$; hence, there is a solution φ_c for Eq. (3.4). Moreover, since $\{0, c_{j\sigma\mu}\}$ is not orthogonal to $\ker(\tilde{\mathcal{L}}(\lambda_s))^*$, it follows that the vector $(c_1, \dots, c_J)$ is nontrivial. Thus, the function V given by (3.3) is not a polynomial in y_1, y_2.

[2] This solution exists and is unique because $-is$ is not an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$.

The main result of this section is as follows.

Theorem 3.1. *Let Conditions* 3.1–3.4 *hold and u be a generalized solution of problem* (2.5), (2.6) *with right-hand side* $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$. *Then* $u \in W^{2m}(G)$.

3.2. Proof of the main result

Let $U_j(y') = u_j(y(y'))$, $j = 1, \dots, N$, be the functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying problem (2.10), (2.11) with right-hand side $\{F_j, \Psi_{j\sigma\mu}\}$ (see Section 2.2).

Set

$$D_\chi = 2\max\{\chi_{j\sigma ks}\}, \qquad d_\chi = \min\{\chi_{j\sigma ks}\}/2. \tag{3.5}$$

Let $\varepsilon > 0$ be so small that $D_\chi \varepsilon < \varepsilon_1$ (where ε and ε_1 are defined in Section 2.1).

Introduce the spaces of vector-valued functions

$$\mathcal{W}^k(K^\varepsilon) = \prod_j W^k(K_j^\varepsilon), \qquad \mathcal{H}_a^k(K^\varepsilon) = \prod_j H_a^k(K_j^\varepsilon), \quad k \geqslant 0; \tag{3.6}$$

$$\mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) = \prod_{j,\sigma} W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon),$$

$$\mathcal{H}_a^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) = \prod_{j,\sigma} H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon). \tag{3.7}$$

Similarly, one can introduce the spaces $\mathcal{W}^k(K)$, $\mathcal{H}_a^k(K)$, $\mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma)$, and $\mathcal{H}_a^{2m-\mathbf{m}-1/2}(\gamma)$.

Since any generalized solution $u \in W^{2m}(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})$ for any $\delta > 0$ by definition, it follows that

$$U_j \in W^{2m}\big(K_j^{\varepsilon_1} \setminus \overline{\mathcal{O}_\delta(0)}\big) \quad \forall \delta > 0. \tag{3.8}$$

It follows from the belonging $U \in \mathcal{W}^\ell(K^{\varepsilon_1})$ that

$$U \in \mathcal{H}_0^0(K^{\varepsilon_1}). \tag{3.9}$$

Further, we have (see (2.10), (2.11)) $\{F_j\} \in \mathcal{W}^0(K^\varepsilon)$ and, by the belonging $f_{i\mu} \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$, by relation (2.7), and by estimate (2.3), we have $\{\Psi_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$. Therefore,

$$\{F_j\} \in \mathcal{H}_{2m}^0(K^\varepsilon), \qquad \{\Psi_{j\sigma\mu}\} \in \mathcal{H}_{2m}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon). \tag{3.10}$$

It follows from relations (3.8)–(3.10) and from Lemma A.5 that

$$U \in \mathcal{H}_{2m}^{2m}(K^{\varepsilon_1}). \tag{3.11}$$

To prove Theorem 3.1, it suffices to show that $U \in \mathcal{W}^{2m}(K^\varepsilon)$.

Lemma 3.1. *Let $U \in \mathcal{W}^\ell(K^\varepsilon)$, U_j satisfy relations* (3.8), *and U be a solution*[3] *of problem* (2.10), (2.11) *with right-hand side $\{F_j, \Psi_{j\sigma\mu}\} \in \mathcal{W}^0(K^\varepsilon) \times \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$. Then*

$$U = Q + \hat{U}, \tag{3.12}$$

where $\hat{U} \in \mathcal{H}^{2m}_{2m-\ell}(K^\varepsilon)$ and $Q = (Q_1, \dots, Q_N)$ is a polynomial vector of degree[4] $\ell - 1$.

Proof. Due to (3.11), it suffices to consider the case $\ell \geqslant 1$. Let δ be an arbitrary number such that $0 < \delta < 1$. By Lemma 4.11 in [19], for each function $\Psi_{j\sigma\mu} \in \mathcal{W}^{2m-m_{j\sigma\mu}-1/2}(\gamma^\varepsilon_{j\sigma})$, there is a polynomial $P_{j\sigma\mu}(r)$ of degree $2m - m_{j\sigma\mu} - 2$ such that

$$\{\Psi_{j\sigma\mu} - P_{j\sigma\mu}\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_{2m-\ell-\delta}\big(\gamma^\varepsilon\big).$$

Using Lemma A.8, one can construct a function

$$W^1 = \sum_{s=0}^{\ell-1} \sum_{l=0}^{l_1} r^s (i \ln r)^l \varphi^1_{sl}(\omega) \in \mathcal{H}^{2m}_{2m}\big(K^\varepsilon\big), \tag{3.13}$$

where $\varphi^1_{sl} \in \prod_j W^{2m}(-\omega_j, \omega_j)$, such that

$$\big\{\mathbf{P}_j(y, D_y) W^1_j\big\} \in \mathcal{H}^0_{2m-\ell-\delta}\big(K^\varepsilon\big), \qquad \big\{\mathbf{B}_{j\sigma\mu}(y, D_y) W^1 - P_{j\sigma\mu}\big\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_{2m-\ell-\delta}\big(\gamma^\varepsilon\big).$$

Therefore, $\{\mathbf{P}_j(y, D_y)(U_j - W^1_j)\} \in \mathcal{H}^0_{2m-\ell-\delta}(K^\varepsilon)$, $\{\mathbf{B}_{j\sigma\mu}(y, D_y)(U_j - W^1)\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_{2m-\ell-\delta}(\gamma^\varepsilon)$.

It follows from (3.11) and (3.13) that $U - W^1 \in \mathcal{H}^{2m}_{2m}(K^\varepsilon)$. Due to Lemma 2.1, we can choose a number δ, $0 < \delta < 1$, in such a way that the band $1 - \ell - \delta \leqslant \operatorname{Im} \lambda < 1 - \ell$ has no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Therefore, applying Lemmas A.7 and A.8, we obtain

$$U - W^1 = W^2 + \hat{U},$$

where

$$W^2 = \sum_{n=1}^{n_0} \sum_{l=0}^{l_2} r^{i\mu_n} (i \ln r)^l \varphi^2_{nl}(\omega),$$

$\{\mu_1, \dots, \mu_{n_0}\}$ is the set of all eigenvalues lying in the band $1 - \ell \leqslant \operatorname{Im} \lambda < 1$ (in fact, we have to consider the eigenvalues in the band $1 - \ell - \delta \leqslant \operatorname{Im} \lambda < 1$, but the band $1 - \ell - \delta \leqslant \operatorname{Im} \lambda < 1 - \ell$ has no eigenvalues by the choice of δ), $\varphi^2_{nl} \in \prod_j W^{2m}(-\omega_j, \omega_j)$, and $\hat{U} \in \mathcal{H}^{2m}_{2m-\ell-\delta}(K^\varepsilon) \subset \mathcal{H}^{2m}_{2m-\ell}(K^\varepsilon)$.

[3] Since $U \in \mathcal{H}^{2m}_{2m}(K^{\varepsilon_1})$ due to (3.11) and $\{F_j, \Psi_{j\sigma\mu}\} \in \mathcal{H}^0_{2m}(K^\varepsilon) \times \mathcal{H}^{2m-\mathbf{m}-1/2}_{2m}(\gamma^\varepsilon)$, relations (2.10), (2.11) can be understood as equalities in the corresponding weighted spaces.

[4] Saying "a polynomial of degree s," we always mean "a polynomial of degree no greater than s." We mean that the polynomial equals zero if $s < 0$.

Since $s \leqslant \ell - 1$ (in the formula for W^1), $\operatorname{Re} i\mu_n \leqslant \ell - 1$ (in the formula for W^2), and $W^1 + W^2 = U - \hat{U} \in \mathcal{W}^\ell(K^\varepsilon)$, it follows from Lemma A.3 that $W^1 + W^2$ is a polynomial vector of degree $\ell - 1$. $\square$

Lemma 3.2. *Let the hypotheses of Lemma* 3.1 *be fulfilled, and let Conditions* 3.2–3.4 *hold. Then*

$$U = W + U' \tag{3.14}$$

where $W = (W_1, \dots, W_N)$ *is a polynomial vector of degree* $2m - 2$, $U' \in \mathcal{H}_\delta^{2m}(K^\varepsilon)$ (δ *is such that* $0 < \delta < 1$ *and the band* $1 - 2m < \operatorname{Im}\lambda \leqslant 1 - 2m + \delta$ *contains no eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$), *and*

$$\begin{gathered}\{\mathbf{P}_j(y, D_y)U'_j\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)U'\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).\end{gathered} \tag{3.15}$$

Proof. 1. Consider the function $\hat{U}$ defined by Lemma 3.1. The function $\hat{U}$ belongs to $\mathcal{H}_{2m-\ell}^{2m}(K^\varepsilon)$, and, by virtue of relations (2.10), (2.11), and (3.12), it is a solution of the problem

$$\begin{aligned}\mathbf{P}_j(y, D_y)\hat{U}_j &= F_j - \mathbf{P}_j(y, D_y)Q_j \qquad (y \in K_j^\varepsilon),\\ \mathbf{B}_{j\sigma\mu}(y, D_y)\hat{U} &= \Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)Q \quad (y \in \gamma_{j\sigma}^\varepsilon).\end{aligned} \tag{3.16}$$

Since $\{F_j\} \in \mathcal{W}^0(K^\varepsilon)$ and Q is a polynomial vector, it follows that

$$\{F_j - \mathbf{P}_j(y, D_y)Q_j\} \in \mathcal{H}_0^0(K^\varepsilon). \tag{3.17}$$

Further, $\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)Q \in W^{2m-m_{j\sigma\mu}-1/2}(\gamma_j^\varepsilon)$. Hence, by Lemma 4.11 in [19], there exists a polynomial $P_{j\sigma\mu}(r)$ of degree $2m - m_{j\sigma\mu} - 2$ such that

$$\{\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)Q - P_{j\sigma\mu}\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \tag{3.18}$$

for any $0 < \delta < 1$. Moreover, since

$$\{\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)Q\} = \{\mathbf{B}_{j\sigma\mu}(y, D_y)\hat{U}\} \in \mathcal{H}_{2m-\ell}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon),$$

we see that each polynomial $P_{j\sigma\mu}(r)$ consists of monomials of degree $\max(0, \ell - m_{j\sigma\mu}), \dots, 2m - m_{j\sigma\mu} - 2$ (the polynomial $P_{j\sigma\mu}(r)$ is absent if $\ell = 2m - 1$).

2. We write each polynomial $P_{j\sigma\mu}(r)$ as follows:

$$P_{j\sigma\mu}(r) = c_{j\sigma\mu} r^{\ell - m_{j\sigma\mu}} + c'_{j\sigma\mu} r^{\ell - m_{j\sigma\mu}+1} + \cdots, \tag{3.19}$$

where, in particular, $c_{j\sigma\mu} = 0$ for all j, σ, μ such that $\ell \leqslant m_{j\sigma\mu} - 1$ (cf. (3.1) for $s = \ell$). Therefore, $\{c_{j\sigma\mu}\} \in C_\ell$.

We consider the auxiliary problem

$$\mathcal{P}_j(D_y)W^\ell = 0, \qquad \mathcal{B}_{j\sigma\mu}(D_y)W^\ell = c_{j\sigma\mu} r^{\ell - m_{j\sigma\mu}}, \tag{3.20}$$

where $\mathcal{P}_j(D_y)$ and $\mathcal{B}_{j\sigma\mu}(D_y)$ are the same as in (2.12). By virtue of Conditions 3.3 and 3.4 (see Remark 3.2), there exists a solution $W^\ell(y)$ of problem (3.20) such that $W^\ell(y)$ is a homogeneous polynomial of degree ℓ.

Using (3.19) and (3.20) and expanding the coefficients of $\mathbf{B}_{j\sigma\mu}(y, D_y)$ by the Taylor formula, we obtain

$$\begin{gathered}\{\mathbf{P}_j(y, D_y)W_j^\ell\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)W^\ell - P_{j\sigma\mu} + P'_{j\sigma\mu}\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon),\end{gathered} \tag{3.21}$$

where $P'_{j\sigma\mu}(r)$ is a polynomial consisting of monomials of degree $\max(0, \ell - m_{j\sigma\mu} + 1), \dots, 2m - m_{j\sigma\mu} - 2$.

It follows from (3.17), (3.18), and (3.21) that

$$\begin{gathered}\{F_j - \mathbf{P}_j(y, D_y)(Q_j + W_j^\ell)\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)(Q + W^\ell) - P'_{j\sigma\mu}\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).\end{gathered} \tag{3.22}$$

3. Repeating the procedure described in item (2) finitely many times (and using Conditions 3.3 and 3.4 each time), we obtain

$$\begin{gathered}\{F_j - \mathbf{P}_j(y, D_y)(Q_j + W_j^\ell + \dots + W_j^{2m-2})\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)(Q + W^\ell + \dots + W^{2m-2})\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon),\end{gathered} \tag{3.23}$$

where W^s is a homogeneous polynomial vector of degree s, $s = \ell, \dots, 2m - 2$ (note that a homogeneous polynomial vector of degree $2m - 1$ already belongs to $\mathcal{H}_\delta^{2m}(K^\varepsilon)$). If $\ell = 2m - 1$, then the polynomials W^s in (3.23) are absent; in this case, the second relation in (3.23) follows from (3.18), where $P_{j\sigma\mu}$ is absent.

Combining (3.16) and (3.23) yields

$$\begin{gathered}\{\mathbf{P}_j(y, D_y)(\hat{U}_j - W_j^\ell - \dots - W_j^{2m-2})\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)(\hat{U} - W^\ell - \dots - W^{2m-2})\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).\end{gathered} \tag{3.24}$$

4. Since the line $\operatorname{Im}\lambda = 1 - 2m + \delta$ has no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ and relations (3.24) hold, it follows from Lemmas A.7, A.8, and Conditions 3.2–3.4 that the function $\hat{U} + W^\ell + \dots + W^{2m-2}$ belongs to the space $\mathcal{H}_\delta^{2m}(K^\varepsilon)$ up to a polynomial consisting of monomials of degree $\min_{s \in i\Lambda} s, \dots, 2m - 2$ (this polynomial is absent if $\ell = 2m - 1$). In other words, there is a polynomial vector $\hat{W}$ consisting of monomials of degree $l, \dots, 2m - 2$ such that

$$\begin{gathered}\hat{U} + \hat{W} \in \mathcal{H}_\delta^{2m}(K^\varepsilon),\\ \{\mathbf{P}_j(y, D_y)(\hat{U}_j + \hat{W}_j)\} \in \mathcal{H}_0^0(K^\varepsilon),\\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)(\hat{U} + \hat{W})\} \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).\end{gathered} \tag{3.25}$$

Now the conclusion of the lemma follows from Lemma 3.1 and from relations (3.25). □

Lemma 3.3. *Let the hypotheses of Lemma* 3.1 *be fulfilled, and let Conditions* 3.1–3.4 *hold. Then* $U \in \mathcal{W}^{2m}(K^\varepsilon)$.

Proof. It follows from (3.15) and from Lemma A.10 that there exists a function $V \in \mathcal{H}_\delta^{2m}(K) \cap \mathcal{W}^{2m}(K)$ such that

$$\begin{aligned} &\{\mathbf{P}_j(y, D_y)(U_j' - V_j)\} \in \mathcal{H}_0^0(K^\varepsilon),\\ &\{\mathbf{B}_{j\sigma\mu}(y, D_y)(U' - V)\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon). \end{aligned} \tag{3.26}$$

Due to (3.26) and the fact that the strip $1 - 2m \leqslant \operatorname{Im}\lambda \leqslant 1 - 2m + \delta$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$, we can use Lemma A.7 to obtain that $U' - V \in \mathcal{H}_0^{2m}(K^\varepsilon) \subset \mathcal{W}^{2m}(K^\varepsilon)$. Combining this relation with Lemma 3.2 completes the proof. □

Theorem 3.1 results from (2.7) and from Lemma 3.3.

4. The border case: Consistency conditions

4.1. Behavior of generalized solutions near the conjugation points

Let Λ be the same set of eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ as in Section 3. In this section, we consider the following condition instead of Condition 3.1.

Condition 4.1. *The line* $\operatorname{Im}\lambda = 1 - 2m$ *contains only the eigenvalue* $\lambda = i(1 - 2m)$ *of the operator* $\tilde{\mathcal{L}}(\lambda)$. *This eigenvalue is a proper one.*

The principal difference between the results of this section and those of Section 3 is related to the behavior of generalized solutions near the set (orbit) $\mathcal{K}$. If Condition 4.1 holds, then Lemma 3.2 remains valid. However, the conclusion of Lemma 3.3 is no longer true because Lemma A.10 is inapplicable when the line $\operatorname{Im}\lambda = 1 - 2m$ contains an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$. In this section, we make use of other results from [14]. To do this, we impose certain consistency conditions on the behavior of the functions $f_{i\mu}$ and the coefficients of nonlocal terms near the set (orbit) $\mathcal{K}$.

Let $\tau_{j\sigma}$ be the unit vector co-directed with the ray $\gamma_{j\sigma}$. Consider the operators

$$\frac{\partial^{2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}U \equiv \frac{\partial^{2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{2m-m_{j\sigma\mu}-1}}\left(\sum_{k,s}\big(B_{j\sigma\mu ks}(D_y)U_k\big)(\mathcal{G}_{j\sigma ks}y)\right).$$

Using the chain rule, we can write

$$\frac{\partial^{2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}U \equiv \sum_{k,s}\big(\hat{B}_{j\sigma\mu ks}(D_y)U_k\big)(\mathcal{G}_{j\sigma ks}y) \tag{4.1}$$

where $\hat{B}_{j\sigma\mu ks}(D_y)$ are some homogeneous differential operators of order $2m-1$ with constant coefficients. Formally replacing the nonlocal operators by the corresponding local operators in (4.1), we introduce the operators

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U \equiv \sum_{k,s} \hat{B}_{j\sigma\mu ks}(D_y)U_k(y). \tag{4.2}$$

If Condition 4.1 holds, then the system of operators (4.2) is linearly dependent (see [14, Section 3.1]). Let

$$\{\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)\} \tag{4.3}$$

be a maximal linearly independent subsystem of system (4.2). In this case, any operator $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)$ which does not enter system (4.3) can be represented as follows:

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y) = \sum_{j',\sigma',\mu'} \beta_{j\sigma\mu}^{j'\sigma'\mu'} \hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y), \tag{4.4}$$

where $\beta_{j\sigma\mu}^{j'\sigma'\mu'}$ are some constants.

Introduce the notion of consistency condition. Let $\{Z_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^{\varepsilon})$ be a vector of functions, each of which is defined on its own interval $\gamma_{j\sigma}^{\varepsilon}$. Consider the functions

$$Z_{j\sigma\mu}^0(r) = Z_{j\sigma\mu}(y)|_{y=(r\cos\omega_j,\, r(-1)^{\sigma}\sin\omega_j)}.$$

Each of the functions $Z_{j\sigma\mu}^0$ belongs to $W^{2m-m_{j\sigma\mu}-1/2}(0,\varepsilon)$.

Definition 4.1. Let $\beta_{j\sigma\mu}^{j'\sigma'\mu'}$ be the constants occurring in (4.4). If the relations

$$\int_0^{\varepsilon} r^{-1} \left| \frac{d^{2m-m_{j\sigma\mu}-1}}{dr^{2m-m_{j\sigma\mu}-1}} Z_{j\sigma\mu}^0 - \sum_{j',\sigma',\mu'} \beta_{j\sigma\mu}^{j'\sigma'\mu'} \frac{d^{2m-m_{j'\sigma'\mu'}-1}}{dr^{2m-m_{j'\sigma'\mu'}-1}} Z_{j'\sigma'\mu'}^0 \right|^2 dr < \infty \tag{4.5}$$

hold for all indices j, σ, μ corresponding to the operators of system (4.2) which do not enter system (4.3), then we say that the *functions $Z_{j\sigma\mu}$ satisfy the consistency condition* (4.5).

Remark 4.1. The relation $\{Z_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^{\varepsilon})$ is sufficient (but not necessary) for the functions $Z_{j\sigma\mu}$ to satisfy the consistency condition (4.5). This follows from Lemma 4.18 in [19].

Now we will show that the following condition is necessary and sufficient for a given generalized solution u to belong to $W^{2m}(G)$.

Condition 4.2. *Let u be a generalized solution of problem* (2.5), (2.6), *$\Psi_{j\sigma\mu}$ the right-hand sides in nonlocal conditions* (2.11), *and W the polynomial vector appearing in Lemma* 3.2. *Then the functions $\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)W$ satisfy the consistency condition* (4.5).

Theorem 4.1. *Let Conditions* 4.1 *and* 3.2–3.4 *hold, and let u be a generalized solution of problem* (2.5), (2.6) *with right-hand side* $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$. *Then* $u \in W^{2m}(G)$ *if and only if Condition* 4.2 *holds.*

Proof. 1. *Necessity.* Let $u \in W^{2m}(G)$. Let the function $U = (U_1, \dots, U_N)$ correspond to the set (orbit) $\mathcal{K}$. Clearly, $U \in \mathcal{W}^{2m}(K^\varepsilon)$. It follows from Lemma 3.2 that $U = W + U'$, where $U' \in \mathcal{H}_\delta^{2m}(K^\varepsilon)$, $0 < \delta < 1$. Since we additionally have $U' = U - W \in \mathcal{W}^{2m}(K^\varepsilon)$, it follows from Sobolev's embedding theorem that $D^\alpha U'(0) = 0$, $|\alpha| \leqslant 2m - 2$. These relations and Lemma A.12 imply that the functions $\Psi_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu} W = \mathbf{B}_{j\sigma\mu}(y, D_y)U'$ satisfy the consistency condition (4.5).

2. *Sufficiency.* Suppose that Condition 4.2 holds. It follows from (3.15) and from Lemma A.11 that there exists a function $V \in \mathcal{H}_\delta^{2m}(K) \cap \mathcal{W}^{2m}(K)$ (δ is the same as in Lemma 3.2) such that

$$\begin{aligned} &\big\{\mathbf{P}_j(y, D_y)\big(U_j' - V_j\big)\big\} \in \mathcal{H}_0^0\big(K^\varepsilon\big), \\ &\big\{\mathbf{B}_{j\sigma\mu}(y, D_y)(U' - V)\big\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big). \end{aligned} \tag{4.6}$$

Due to (4.6) and the fact that the strip $1 - 2m \leqslant \operatorname{Im}\lambda \leqslant 1 - 2m + \delta$ contains only the proper eigenvalue $i(1 - 2m)$ of $\tilde{\mathcal{L}}(\lambda)$, we can use Lemma A.9 to obtain that all the derivatives of order $2m$ of the function $U' - V$ belong to $\mathcal{W}^0(K^\varepsilon)$. It follows from this fact and from the relations

$$U' - V \in \mathcal{H}_\delta^{2m}\big(K^\varepsilon\big) \subset \mathcal{H}_0^{2m-1}\big(K^\varepsilon\big) \subset \mathcal{W}^{2m-1}\big(K^\varepsilon\big)$$

that $U' - V \in \mathcal{W}^{2m}(K^\varepsilon)$. Combining this relation with Lemma 3.2, we complete the proof of the sufficiency part. □

Note that Theorem 4.1 enables us to conclude whether or not a *given* solution u is smooth near the set $\mathcal{K}$, provided that we know the asymptotics for u of the kind (3.14) near the set $\mathcal{K}$ (i.e., if we know the polynomial vector W). Theorem 4.1 shows what affects the smoothness of solutions in principle. Below, this will enable us to obtain a constructive condition which is necessary and sufficient for *any* generalized solution to belong to $W^{2m}(G)$.

4.2. Problem with nonhomogeneous nonlocal conditions

First of all, we show that the right-hand sides $f_{i\mu}$ in nonlocal conditions (2.6) must satisfy a certain consistency condition in order that generalized solutions be smooth.

Denote by $\mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$ the set of functions $\{f_{i\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$ such that the functions $F_{j\sigma\mu}$ (see (2.9)) satisfy the consistency condition (4.5). It follows from Lemma 3.2 in [14] that the set $\mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$ is not closed in the space $\mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$.

Theorem 4.2. *Let Conditions* 4.1 *and* 3.2–3.4 *hold. Then there exist a function* $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$, $\{f_{i\mu}\} \notin \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, *and a function* $u \in W^{2m-1}(G)$ *such that u is a generalized solution of problem* (2.5), (2.6) *with the right-hand side* $\{f_0, f_{i\mu}\}$ *and* $u \notin W^{2m}(G)$.

To prove Theorem 4.2, we preliminarily establish an auxiliary result. Set

$$\varepsilon' = d_\chi \min(\varepsilon, \varkappa_2), \tag{4.7}$$

where d_χ is defined in (3.5).

Lemma 4.1. *Let Condition* 4.1 *hold and a function* $\{Z_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$ *be such that* $\operatorname{supp}\{Z_{j\sigma\mu}\} \subset \mathcal{O}_{\varepsilon/2}(0)$, $\frac{\partial^\beta}{\partial\tau_{j\sigma}^\beta} Z_{j\sigma\mu}(0) = 0$, $\beta \leqslant 2m - m_{j\sigma\mu} - 2$, *and the functions* $Z_{j\sigma\mu}$ *do not satisfy the consistency condition* (4.5). *Then there exists a function* $U \in \mathcal{H}_\delta^{2m}(K) \subset \mathcal{W}^{2m-1}(K)$, $\delta > 0$ *is arbitrary, such that* $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$, $U \notin \mathcal{W}^{2m}(K^\varepsilon)$, *and* U *satisfies the relations*

$$\{\mathbf{P}_j(y, D_y)U_j\} \in \mathcal{W}^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma\mu}(y, D_y)U - Z_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon). \tag{4.8}$$

Proof. 1. By Lemma A.4, there exists a sequence of functions $\{Z_{j\sigma\mu}^n\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma)$, $n = 1, 2, \ldots,$ such that $\operatorname{supp} Z_{j\sigma\mu}^n \subset \mathcal{O}_\varepsilon(0)$, $Z_{j\sigma\mu}^n$ vanish near the origin (hence, they satisfy the consistency condition (4.5)), and $\{Z_{j\sigma\mu}^n\} \to \{Z_{j\sigma\mu}\}$ in $W^{2m-\mathbf{m}-1/2}(\gamma)$. Taking into account Lemma A.1, we also see that $\{Z_{j\sigma\mu}^n\} \to \{Z_{j\sigma\mu}\}$ in $H_\delta^{2m-\mathbf{m}-1/2}(\gamma)$, $\delta > 0$ is arbitrary. Lemma 3.5 in [14] ensures the existence of a sequence $V^n = (V_1^n, \ldots, V_N^n)$ satisfying the following conditions: $V^n \in \mathcal{W}^{2m}(K^d) \cap \mathcal{H}_\delta^{2m}(K^d)$ for any $d > 0$,

$$\mathcal{P}_j(D_y)V_j^n = 0 \quad (y \in K_j), \qquad \mathcal{B}_{j\sigma\mu}(D_y)V^n = Z_{j\sigma\mu}^n(y) \quad (y \in \gamma_{j\sigma}), \tag{4.9}$$

and the sequence V^n converges to a function $V \in \mathcal{H}_\delta^{2m}(K^d)$ in $\mathcal{H}_\delta^{2m}(K^d)$ for any $d > 0$. Passing to the limit in (4.9) (in the spaces $\mathcal{H}_\delta^0(K^d)$ and $\mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(K^d)$, respectively), we obtain

$$\mathcal{P}_j(D_y)V_j = 0 \quad (y \in K_j), \qquad \mathcal{B}_{j\sigma\mu}(D_y)V = Z_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}). \tag{4.10}$$

Consider a cut-off function $\xi \in C_0^\infty(\mathcal{O}_{\varepsilon'}(0))$ equal to one near the origin. Set $U = \xi V$. Clearly, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$ and

$$U \in \mathcal{H}_\delta^{2m}(K) \subset \mathcal{W}^{2m-1}(K).$$

2. We claim that U is the desired function. Indeed, using Leibniz' formula, relations (4.10) and Lemma A.2, we infer (4.8).

It remains to prove that $U \notin \mathcal{W}^{2m}(K^\varepsilon)$. Assume the contrary. Let $U \in \mathcal{W}^{2m}(K^\varepsilon)$. In this case, it follows from Sobolev's embedding theorem and from the belonging $U \in \mathcal{H}_\delta^{2m}(K^\varepsilon)$ ($\delta > 0$ is arbitrary) that $D^\alpha U(0) = 0$, $|\alpha| \leqslant 2m - 2$. Combining this fact with Lemma A.12 implies that the functions $\mathbf{B}_{j\sigma\mu}(y, D_y)U$ satisfy the consistency condition (4.5). However, the functions $\mathbf{B}_{j\sigma\mu}(y, D_y)U - Z_{j\sigma\mu}$ do not satisfy the consistency condition (4.5) in that case. This contradicts (4.8) (see Remark 4.1). $\square$

Proof of Theorem 4.2. 1. We will construct a generalized solution $u \notin W^{2m}(G)$ supported near the set $\mathcal{K}$ so that $\mathbf{B}_{i\mu}^2 u = 0$ due to (2.3).

It was shown in the course of the proof of Lemma 3.2 in [14] that there exists a function $\{Z_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma)$ such that $\operatorname{supp} Z_{j\sigma\mu} \subset \mathcal{O}_{\varepsilon/2}(0)$, $\frac{\partial^\beta}{\partial\tau_{j\sigma}^\beta} Z_{j\sigma\mu}(0) = 0$, $\beta \leqslant 2m - m_{j\sigma\mu} - 2$, and the functions $Z_{j\sigma\mu}$ do not satisfy the consistency condition (4.5). By Lemma 4.1, there exists a function $U \in \mathcal{H}_\delta^{2m}(K) \subset \mathcal{W}^{2m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$, $U \notin \mathcal{W}^{2m}(K)$, and U satisfies relations (4.8). Therefore, $\{\mathbf{P}_j(y, D_y)U_j\} \in \mathcal{W}^0(K^\varepsilon)$, $\{\mathbf{B}_{j\sigma\mu}(y, D_y)U\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$, and the functions $\mathbf{B}_{j\sigma\mu}(y, D_y)U$ do not satisfy the consistency condition (4.5).

2. Introduce a function $u(y)$ such that $u(y) = U_j(y'(y))$ for $y \in \mathcal{O}_{\varepsilon'}(g_j)$ and $u(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_j)$ is the change of variables inverse to the change of variables $y \mapsto y'(g_j)$ from Section 2.1. Since $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$, it follows that $\mathbf{B}^2_{i\mu} u = 0$. Therefore, $u(y)$ is the desired generalized solution of problem (2.5), (2.6). $\square$

Theorem 4.2 shows that if one wants that *any* generalized solution of problem (2.5), (2.6) be smooth, then one must take right-hand sides $\{f_0, f_{i\mu}\}$ from the space $L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$.

Let v be an arbitrary function from the space $W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. Consider the change of variables $y \mapsto y'(g_j)$ from Section 2.1 and introduce the functions

$$B^v_{j\sigma\mu}(y') = \big(\mathbf{B}^2_{i\mu} v\big)\big(y(y')\big), \quad y' \in \gamma^\varepsilon_{j\sigma} \tag{4.11}$$

(cf. (2.9)). We prove that the following condition is necessary and sufficient for any generalized solution to be smooth.

Condition 4.3. (1) *For any $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, the functions $B^v_{j\sigma\mu}$ satisfy the consistency condition* (4.5).

(2) *For any polynomial vector W of degree $2m-2$ the functions $\mathbf{B}_{j\sigma\mu}(y, D_y)W$ satisfy the consistency condition* (4.5).

Note that the validity of Condition 4.3, unlike Condition 4.2, does not depend on a generalized solution. It depends only on the operators $\mathbf{B}^1_{i\mu}$ and $\mathbf{B}^2_{i\mu}$ and on the geometry of the domain G near the set (orbit) $\mathcal{K}$. This is quite natural because we study the smoothness of *all* generalized solutions in this section (while in Section 4.1, we have investigated the smoothness of a fixed solution).

Theorem 4.3. *Let Conditions* 4.1 *and* 3.2–3.4 *hold.*

(1) *If Condition* 4.3 *holds and u is a generalized solution of problem* (2.5), (2.6) *with right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, then $u \in W^{2m}(G)$.*
(2) *If Condition* 4.3 *fails, then there exists a right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$ and a generalized solution u of problem* (2.5), (2.6) *such that $u \notin W^{2m}(G)$.*

Proof. 1. *Sufficiency.* Let Condition 4.3 hold, and let u be an arbitrary generalized solution of problem (2.5), (2.6) with right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$. By (2.7), we have $u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. Therefore, by Condition 4.3, the functions $B^u_{j\sigma\mu}$ satisfy the consistency condition (4.5). Let W be a polynomial vector of degree $2m-2$ defined by Lemma 3.2. Using Condition 4.3 again, we see that the functions $\mathbf{B}_{j\sigma\mu}(y, D_y)W$ satisfy the consistency condition (4.5). Since $\{f_{i\mu}\} \in \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, it follows that the functions $F_{j\sigma\mu}$ satisfy the consistency condition (4.5). Therefore, the functions $\Psi_{j\sigma\mu} = F_{j\sigma\mu} - B^u_{j\sigma\mu}$ and $\mathbf{B}_{j\sigma\mu}(y, D_y)W$ satisfy Condition 4.2. Applying Theorem 4.1, we obtain $u \in W^{2m}(G)$.

2. *Necessity.* Let Condition 4.3 fail. In this case, there exist a function $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and a polynomial vector $W = (W_1, \dots, W_N)$ of degree $2m-2$ such that the functions $B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}W$ do not satisfy the consistency condition (4.5) (one can assume that either $v = 0$, $W \neq 0$ or $v \neq 0$, $W = 0$). Extend the function v to the domain G in such a way that $v(y) = 0$ for $y \in \mathcal{O}_{\varkappa_1/2}(\mathcal{K})$ and $v \in W^{2m}(G)$.

By Lemma 4.11 in [19], there exist polynomials $F'_{j\sigma\mu}(r)$ of degree $2m - m_{j\sigma\mu} - 2$ such that

$$\{B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W - F'_{j\sigma\mu}\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_\delta(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon),$$

where $\delta > 0$ is arbitrary. Hence,

$$\frac{\partial^\beta}{\partial\tau^\beta_{j\sigma}}\big(B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W - F'_{j\sigma\mu}\big)(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2.$$

Since $\frac{d^{2m-m_{j\sigma\mu}-1}}{dr^{2m-m_{j\sigma\mu}-1}}F'_{j\sigma\mu}(r) \equiv 0$, it follows that the functions $F'_{j\sigma\mu}$ satisfy the consistency condition (4.5). Therefore, the functions $B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W - F'_{j\sigma\mu}$ do not satisfy the consistency condition (4.5).

By Lemma 4.1, there exists a function $U' \in \mathcal{H}^{2m}_\delta(K) \subset \mathcal{W}^{2m-1}(K)$ such that $\operatorname{supp} U' \subset \mathcal{O}_{\varepsilon'}(0)$, $U' \notin \mathcal{W}^{2m}(K^\varepsilon)$, and

$$\begin{gathered}\{\mathbf{P}_j(y, D_y)U'_j\} \in \mathcal{W}^0(K^\varepsilon),\\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)U' - \big(F'_{j\sigma\mu} - B^v_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)W\big)\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_0(\gamma^\varepsilon).\end{gathered} \tag{4.12}$$

One can also write the latter relation as follows:

$$\{\mathbf{B}_{j\sigma\mu}(y, D_y)(U' + W) + B^v_{j\sigma\mu} - F'_{j\sigma\mu}\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_0(\gamma^\varepsilon). \tag{4.13}$$

Introduce a function $u'(y)$ such that $u'(y) = U'_j(y'(y)) + \xi_j(y)W_j$ for $y \in \mathcal{O}_{\varepsilon'}(g_j)$ and $u'(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_j)$ is the change of variables inverse to the change of variables $y \mapsto y'(g_j)$ from Section 2.1, while $\xi_j \in C^\infty_0(O_{\varepsilon'}(g_j))$, $\xi_j(y) = 1$ for $y \in \mathcal{O}_{\varepsilon'/2}(g_j)$, and ε' is given by (4.7). Let us prove that the function $u = u' + v$ is the desired one. Clearly, $u \in W^{2m-1}(G)$, $u \notin W^{2m}(G)$, and u satisfies relations (2.7). It follows from the belonging $v \in W^{2m}(G)$ and from relations (4.12) that

$$\mathbf{P}(y, D_y)u \in L_2(G).$$

Consider the functions $f_{i\mu} = \mathbf{B}^0_{i\mu}u + \mathbf{B}^1_{i\mu}u + \mathbf{B}^2_{i\mu}u$. It follows from the belonging $v \in W^{2m}(G)$, from relations (2.7), and from inequality (2.3) that $f_{i\mu} \in W^{2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})$ for any $\delta > 0$. Consider the behavior of $f_{i\mu}$ near the set $\mathcal{K}$. Note that $\mathbf{B}^2_{i\mu}u' = 0$ by (2.3). Furthermore, $\mathbf{B}^0_{i\mu}v + \mathbf{B}^1_{i\mu}v = 0$ for $y \in \mathcal{O}_{\varkappa_1/D_\chi}(\mathcal{K})$. Therefore,

$$f_{i\mu} = \mathbf{B}^0_{i\mu}u' + \mathbf{B}^1_{i\mu}u' + \mathbf{B}^2_{i\mu}v \quad \big(y \in \mathcal{O}_{\varkappa_1/D_\chi}(\mathcal{K})\big). \tag{4.14}$$

Introduce the functions $F_{j\sigma\mu}(y') = f_{i\mu}(y(y'))$, where $y \mapsto y'(g_j)$ is the change of variables from Section 2.1. It follows from (4.14) and from (4.13) that $\{F_{j\sigma\mu} - F'_{j\sigma\mu}\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_0(\gamma^\varepsilon)$. Therefore, $\{F_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$ and the functions $F_{j\sigma\mu}$, together with $F'_{j\sigma\mu}$, satisfy the consistency condition (4.5). Hence $\{f_{i\mu}\} \in \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, which completes the proof. $\square$

4.3. Problem with regular nonlocal conditions

Definition 4.2. We say that a function $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ is *admissible* if there exists a polynomial vector $W = (W_1, \dots, W_N)$ of degree $2m-2$ such that

$$\frac{\partial^\beta}{\partial \tau_{j\sigma}^\beta}\big(B_{j\sigma\mu}^v + \mathbf{B}_{j\sigma\mu}(y, D_y)W\big)(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2. \tag{4.15}$$

Any polynomial vector W of degree $2m-2$ satisfying relations (4.15) is said to be an *admissible polynomial vector corresponding to the function* v.

Let τ_{gi} be the unit vector parallel to Γ_i near the point $g \in \overline{\Gamma_i} \cap \mathcal{K}$.

Definition 4.3. (1) The right-hand sides $f_{i\mu}$ in nonlocal conditions (2.6) are said to be *regular* if $\{f_{i\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$ and

$$\frac{\partial^\beta}{\partial \tau_{gi}^\beta} f_{i\mu}(g) = 0, \quad \beta \leqslant 2m - m_{i\mu} - 2, \ g \in \overline{\Gamma_i} \cap K.$$

(2) The right-hand sides $\Psi_{j\sigma\mu}$ in nonlocal conditions (2.11) are said to be *regular* if $\{\Psi_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$ and

$$\frac{\partial^\beta}{\partial \tau_{j\sigma}^\beta}\Psi_{j\sigma\mu}(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2.$$

If $m_{i\mu} = 2m-1$ or $m_{j\sigma\mu} = 2m-1$, then the corresponding relations are absent.

In particular, the right-hand sides $\{f_{i\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\partial G)$ and $\{\Psi_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$ are regular due to Sobolev's embedding theorem. In this subsection, we prove that the following condition (which is weaker than Condition 4.3) is necessary and sufficient for any generalized solution of problem (2.5), (2.6) with regular right-hand sides $\{f_{i\mu}\} \in \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$ to be smooth.

Condition 4.4. *For each admissible function v and each admissible polynomial vector W (of degree $2m-2$) corresponding to v, the functions $B_{j\sigma\mu}^v + \mathbf{B}_{j\sigma\mu}(y, D_y)W$ satisfy the consistency condition* (4.5).

Theorem 4.4. *Let Conditions* 4.1 *and* 3.2–3.4 *hold.*

(1) *If Condition* 4.4 *holds and u is a generalized solution of problem* (2.5), (2.6) *with right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, where $f_{i\mu}$ are regular, then $u \in W^{2m}(G)$.*
(2) *If Condition* 4.4 *fails, then there exists a right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\partial G)$ and a generalized solution u of problem* (2.5), (2.6) *such that $u \notin W^{2m}(G)$.*

Proof. 1. *Sufficiency.* Let Condition 4.4 hold, and let u be an arbitrary generalized solution of problem (2.5), (2.6) with right-hand side $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$, where $f_{i\mu}$ are regular. By (2.7), we have $u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$.

It follows from the properties of $f_{i\mu}$ that the right-hand sides in nonlocal conditions (2.11) have the form

$$\Psi_{j\sigma\mu} = F_{j\sigma\mu} - B^u_{j\sigma\mu}, \tag{4.16}$$

where $\{F_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$,

$$\frac{\partial^\beta}{\partial \tau^\beta_{j\sigma}} F_{j\sigma\mu}(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2, \tag{4.17}$$

and $F_{j\sigma\mu}$ satisfy the consistency condition (4.5).

Further, let $U = W + U'$, where $U' \in \mathcal{H}^{2m}_\delta(K^\varepsilon)$ and W are the function and the polynomial vector (of degree $2m-2$) defined in Lemma 3.2. It follows from (2.11) and (4.16) that

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U' = F_{j\sigma\mu} - \big(B^u_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W\big).$$

Since $\{B^u_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W - F_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$ and $U' \in \mathcal{H}^{2m}_\delta(K^\varepsilon)$, it follows that

$$\begin{aligned}&\big\{B^u_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W - F_{j\sigma\mu}\big\}\\ &\quad = \big\{-\mathbf{B}_{j\sigma\mu}(y, D_y)U'\big\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_\delta\big(\gamma^\varepsilon\big) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big).\end{aligned}$$

It follows from this relation and from (4.17) that

$$\frac{\partial^\beta}{\partial \tau^\beta_{j\sigma}} \big(B^u_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W\big)(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2,$$

i.e., u is an admissible function and W is an admissible polynomial vector corresponding to u. Hence, by virtue of (4.16) and by Condition 4.4, Condition 4.2 holds. Combining this fact with Theorem 4.1 implies $u \in W^{2m}(G)$.

2. *Necessity.* Let Condition 4.4 fail. In this case, there exists a function $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and a polynomial vector $W = (W_1, \dots, W_N)$ of degree $2m-2$ such that

$$\frac{\partial^\beta}{\partial \tau^\beta_{j\sigma}} \big(B^u_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W\big)(0) = 0, \quad \beta \leqslant 2m - m_{j\sigma\mu} - 2,$$

and the functions $B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W$ do not satisfy the consistency condition (4.5).

We must find a function $u \in W^\ell(G)$ satisfying relations (2.7) such that $u \notin W^{2m}(G)$ and

$$\mathbf{P}(y, D_y)u \in L_2(G), \qquad \big\{\mathbf{B}^0_{i\mu}u + \mathbf{B}^1_{i\mu}u + \mathbf{B}^2_{i\mu}u\big\} \in \mathcal{H}^{2m-\mathbf{m}-1/2}_0(\partial G).$$

To do this, one can repeat the proof of assertion (2) of Theorem 4.3, assuming that v is the above function, W is the above polynomial vector, and $F'_{j\sigma\mu}(y) \equiv 0$ (which is possible due to the relation $B^v_{j\sigma\mu} + \mathbf{B}_{j\sigma\mu}(y, D_y)W \in \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon)$, where $\delta > 0$ is arbitrary). $\square$

5. Violation of smoothness of generalized solutions

5.1. Violation of Conditions 3.1 and 4.1 or Condition 3.2

The title of this subsection means that the following condition holds.

Condition 5.1. *The band* $1 - 2m \leqslant \operatorname{Im}\lambda < 1 - \ell$ *contains an improper eigenvalue of the operator* $\tilde{\mathcal{L}}(\lambda)$.

We show that the smoothness of generalized solutions can be violated for any operators $\mathbf{B}^2_{i\mu}$.

Theorem 5.1. *Let Condition* 5.1 *hold. Then there exists a right-hand side* $\{f_0, f_{i\mu}\} \in L_2(G) \times \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\partial G)$ *and a generalized solution* u *of problem* (2.5), (2.6) *such that* $u \notin W^{2m}(G)$.

Proof. 1. Let $\lambda = \lambda_0$ be an improper eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, $1 - 2m \leqslant \operatorname{Im}\lambda_0 < 1 - \ell$. Consider the function

$$V = r^{i\lambda_0} \sum_{l=0}^{l_0} \frac{1}{l!}(i \ln r)^l \varphi^{(l_0-l)}(\omega) \in \mathcal{W}^\ell\big(K^d\big) \quad \forall d > 0, \tag{5.1}$$

where $\varphi^{(0)}, \dots, \varphi^{(\varkappa-1)}$ are an eigenvector and associated vectors (a Jordan chain of length $\varkappa \geqslant 1$) of the operator $\tilde{\mathcal{L}}(\lambda)$ corresponding to the eigenvalue λ_0. The number l_0 ($0 \leqslant l_0 \leqslant \varkappa - 1$) occurring in the definition of V is such that the function V is not a polynomial vector in y_1, y_2. Such a number l_0 does exist because λ_0 is not a proper eigenvalue (if $\operatorname{Im}\lambda$ is a noninteger or $\operatorname{Im}\lambda$ is an integer but $\operatorname{Re}\lambda \neq 0$, then we can take $l_0 = 0$).

Since V is not a polynomial vector, it follows from Lemma A.3 that

$$V \notin \mathcal{W}^{2m}\big(K^d\big) \quad \forall d > 0. \tag{5.2}$$

It follows from Lemma A.6 that

$$\mathcal{P}_j(D_y)V_j = 0, \qquad \mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} = 0. \tag{5.3}$$

Using (5.3) and the Taylor expansion for the coefficients of $\mathbf{P}_j(y, D_y)$ and $\mathbf{B}_{j\sigma\mu}(y, D_y)$, we have

$$\big\{\mathbf{P}_j(y, D_y)V_j - P_j\big\} \in \mathcal{W}^0\big(K^\varepsilon\big), \qquad \big\{\mathbf{B}_{j\sigma\mu}(y, D_y)V - P_{j\sigma\mu}\big\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big), \tag{5.4}$$

where P_j is a linear combination of terms of the kind

$$r^{i\lambda_0-2m+1}(i\ln r)^l\varphi(\omega), \dots, r^{i\lambda_0-2m+k_0}(i\ln r)^l\varphi(\omega),$$

$P_{j\sigma\mu}$ is a linear combination of terms of the kind

$$r^{i\lambda_0-m_{j\sigma\mu}+1}(i\ln r)^l,\ldots,r^{i\lambda_0-m_{j\sigma\mu}+k_0}(i\ln r)^l,$$

$\varphi(\omega)$ are infinitely smooth vector-valued functions, and $k_0\in\mathbb{N}$ is such that

$$-\operatorname{Im}\lambda_0-2m+k_0\leqslant-1,\qquad -\operatorname{Im}\lambda_0-2m+k_0+1>-1. \tag{5.5}$$

Clearly, one can set $P_j=0$ and $P_{j\sigma\mu}=0$ if inequalities (5.5) are true for $k_0=0$, i.e., if $1-2m\leqslant \operatorname{Im}\lambda_0<2-2m$.

Using Lemma A.8, we can construct the function

$$V'=\sum_{k=1}^{k_0}\sum_{l=0}^{l'}r^{i\lambda_0+k}(i\ln r)^{l_k}\varphi_{kl}(\omega)\in W^{\ell}\big(K^d\big)\quad\forall d>0 \tag{5.6}$$

such that

$$\big\{\mathbf{P}_j(y,D_y)V'_j-P_j\big\}\in\mathcal{W}^0\big(K^\varepsilon\big),\qquad \big\{\mathbf{B}_{j\sigma\mu}(y,D_y)V'-P_{j\sigma\mu}\big\}\in\mathcal{H}_0^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big). \tag{5.7}$$

Consider a cut-off function $\xi\in C_0^\infty(\mathcal{O}_{\varepsilon'}(0))$ equal to one near the origin, where ε' is given by (4.7). Set $U=\xi(V-V')$. Clearly, $\operatorname{supp}U\subset\mathcal{O}_{\varepsilon'}(0)$; hence,

$$\operatorname{supp}\mathbf{B}_{j\sigma\mu}(y,D_y)U\subset\overline{\gamma_{j\sigma}}\cap\mathcal{O}_{\varkappa_2}(0). \tag{5.8}$$

It follows from (5.1), (5.6), and (5.2) that

$$U\in\mathcal{W}^{\ell}(K),\qquad U\notin\mathcal{W}^{2m}\big(K^d\big)\quad\forall d>0. \tag{5.9}$$

Moreover, by virtue of (5.4) and (5.7), we have

$$\big\{\mathbf{P}_j(y,D_y)U_j\big\}\in\mathcal{W}^0\big(K^\varepsilon\big),\qquad \big\{\mathbf{B}_{j\sigma\mu}(y,D_y)U\big\}\in\mathcal{H}_0^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big). \tag{5.10}$$

2. Consider the function $u(y)$ given by $u(y)=U_j(y'(y))$ for $y\in\mathcal{O}_{\varepsilon'}(g_j)$ and $u(y)=0$ for $y\notin\mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y'\mapsto y(g_j)$ is the change of variables inverse to the change of variables $y\mapsto y'(g_j)$ from Section 2.1. The function u is the desired one. Indeed, $u\notin W^{2m}(G)$ due to (5.9). Furthermore, $\mathbf{B}^2_{i\mu}u=0$ due to inequality (2.3) because $\operatorname{supp}u\subset\mathcal{O}_{\varkappa_1}(\mathcal{K})$. It follows from the equality $\mathbf{B}^2_{i\mu}u=0$ and from relations (5.10) that the function u satisfies the following relations:

$$\mathbf{P}(y,D_y)u\in L_2(G),\qquad \mathbf{B}^0_{i\mu}u+\mathbf{B}^1_{i\mu}u+\mathbf{B}^2_{i\mu}u\in H_0^{2m-m_{i\mu}-1/2}(\Gamma_i),$$
$$\operatorname{supp}\big(\mathbf{B}^0_{i\mu}u+\mathbf{B}^1_{i\mu}u+\mathbf{B}^2_{i\mu}u\big)\subset\overline{\Gamma_i}\cap\mathcal{O}_{\varkappa_2}(\mathcal{K}).\qquad\square \tag{5.11}$$

5.2. Violation of Condition 3.3 or 3.4

If $\ell = 2m - 1$, then all the possibilities for the location of eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ have been investigated. It remains to assume that $\ell \leqslant 2m - 2$ and Condition 3.3 or 3.4 fails.

Theorem 5.2. *Suppose that Condition* 3.2 *holds while Condition* 3.3 *or* 3.4 *fails. Then there is a right-hand side* $\{f_0, f^1_{i\mu} + f^2_{i\mu}\} \in L_2(G) \times \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$ *and a generalized solution u of problem* (2.5), (2.6) *such that* $u \notin W^{2m}(G)$, *where* $f^1_{i\mu}$ *is a polynomial of degree* $2m - m_{i\mu} - 2$ *in a neighborhood of the point* $g \in \overline{\Gamma_i} \cap \mathcal{K}$ *and* $\{f^2_{i\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\partial G)$.

Proof. 1. Due to part (2) of Remark 3.2, there is a function V given by (3.3) such that

$$V \in \mathcal{W}^{\ell}(K^d), \qquad V \notin \mathcal{W}^{2m}(K^d) \quad \forall d > 0, \tag{5.12}$$

$$\mathcal{P}_j(D_y)V_j = 0, \qquad \mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} = c_{j\sigma\mu} r^{s-m_{j\sigma\mu}} \tag{5.13}$$

for some $s \in \{\ell, \dots, 2m-2\}$ and some (nontrivial) numerical vector $\{c_{j\sigma\mu}\} \in C_s$.

Using (5.13) and the Taylor expansion for the coefficients of $\mathbf{P}_j(y, D_y)$ and $\mathbf{B}_{j\sigma\mu}(y, D_y)$, we have

$$\begin{gathered}\{\mathbf{P}_j(y, D_y)V_j - P_j\} \in \mathcal{W}^0(K^\varepsilon), \\ \{\mathbf{B}_{j\sigma\mu}(y, D_y)V - c_{j\sigma\mu} r^{s-m_{j\sigma\mu}} - P_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon),\end{gathered} \tag{5.14}$$

where the functions P_j and $P_{j\sigma\mu}$ are of the same form as in (5.4).

As in the proof of Theorem 5.1, we can construct a function V' of the form (5.6) (with $i\lambda_0$ replaced by s) satisfying relations (5.7).

Consider a cut-off function $\xi \in C_0^\infty(\mathcal{O}_{\varepsilon'}(0))$ equal to one near the origin, where ε' is given by (4.7). Set $U = \xi(V - V')$. Clearly, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$ and

$$U \in \mathcal{W}^{\ell}(K), \qquad U \notin \mathcal{W}^{2m}(K^d) \quad \forall d > 0. \tag{5.15}$$

Moreover, by virtue of (5.14) and (5.7), we have

$$\{\mathbf{P}_j(y, D_y)U_j\} \in \mathcal{W}^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma\mu}(y, D_y)U - c_{j\sigma\mu} r^{s-m_{j\sigma\mu}}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon). \tag{5.16}$$

We note that, since $\{c_{j\sigma\mu}\} \in C_s$, the function $c_{j\sigma\mu} r^{s-m_{j\sigma\mu}}$ either equals zero (which, in particular, holds for $(j, \sigma, \mu) \in J_s$) or is a monomial of degree $s - m_{j\sigma\mu}$ (i.e., no greater than $2m - m_{j\sigma\mu} - 2$).

2. Consider the function $u(y)$ given by $u(y) = U_j(y'(y))$ for $y \in \mathcal{O}_{\varepsilon'}(g_j)$ and $u(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_j)$ is the change of variables inverse to the change of variables $y \mapsto y'(g_j)$ from Section 2.1. The function u is the desired one. Indeed, $u \notin W^{2m}(G)$ due to (5.15). Furthermore, $\mathbf{B}^2_{i\mu} u = 0$ due to inequality (2.3) because $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. It follows from the equality $\mathbf{B}^2_{i\mu} u = 0$ and from relations (5.16) that the function u satisfies the following relations:

$$\mathbf{P}(y, D_y)u \in L_2(G), \qquad \mathbf{B}^0_{i\mu} u + \mathbf{B}^1_{i\mu} u + \mathbf{B}^2_{i\mu} u = f^1_{i\mu} + f^2_{i\mu},$$

where $f^1_{i\mu}$ is a polynomial[5] of degree no greater than $2m - m_{i\mu} - 2$ in a neighborhood of the point $g \in \overline{\Gamma_i} \cap \mathcal{K}$ and $f^2_{i\mu} \in H_0^{2m-m_{i\mu}-1/2}(\Gamma_i)$. $\square$

Remark 5.1. We remind that the space $\mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$ was introduced in Section 4.2 in the case where the line $\operatorname{Im}\lambda = 1-2m$ contains only the proper eigenvalue $i(1-2m)$. In this case, it was proved in Theorem 4.2 that the smoothness of generalized solutions may violate if the right-hand side $\{f_{i\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\partial G)$ does not belong to $\mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$. Theorem 5.2 shows that if Condition 3.3 or Condition 3.4 fails, then the smoothness of generalized solutions may violate even for the right-hand side $\{f_{i\mu}\} \in \mathcal{S}^{2m-\mathbf{m}-1/2}(\partial G)$.

On the other hand, it is on principle that the smoothness violation in Theorem 5.2 occurs for a nonzero (and even nonregular) right-hand side $\{f_{i\mu}\}$. It can be proved that if we confine ourselves with regular right-hand sides, then Conditions 3.3 and 3.4 are not necessary for the preservation of smoothness.

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Appendix A

This appendix is included for the reader's convenience. Here we collect some known results on weighted spaces and properties of nonlocal operators, which are most frequently referred to in the main part of the paper.

A.1. Some properties of Sobolev and weighted spaces

In this subsection, we formulate some results concerning properties of weighted spaces introduced in Section 2.1. Set

$$K = \{y \in \mathbb{R}^2\colon\ r > 0,\ |\omega| < \omega_0\},$$
$$\gamma_\sigma = \{y \in \mathbb{R}^2\colon\ r > 0,\ \omega = (-1)^\sigma \omega_0\} \quad (\sigma = 1, 2).$$

Lemma A.1. *(See Lemma 2.1 in [14].) Let $\psi \in W^{k-1/2}(\gamma_\sigma)$ ($\sigma = 1$ or 2, $k \geqslant 2$), $\operatorname{supp}\psi \subset \{0 \leqslant r \leqslant \varepsilon\}$ for some $\varepsilon > 0$, and*

$$\frac{d^s}{dr^s}\psi(0) = 0, \quad s = 0, \dots, k-2.$$

[5] The function $f^1_{i\mu}$ (being written in the system of coordinates originated at the point $g \in \overline{\Gamma_i} \cap \mathcal{K}$) either equals zero or is a monomial of degree $s - m_{j\sigma\mu}$.

Then $\psi \in H_\delta^{k-1/2}(\gamma_\sigma)$ *for any* $\delta > 0$ *and*

$$\|\psi\|_{\psi \in H_\delta^{k-1/2}(\gamma_\sigma)} \leqslant c\|\psi\|_{W^{k-1/2}(\gamma_\sigma)},$$

where $c = c(\varepsilon, \delta) > 0$ *does not depend on* ψ.

Lemma A.2. *(See Lemma 3.3′ in [19].) Let a function* $u \in H_a^k(K)$, *where* $k \geqslant 0$ *and* $a \in \mathbb{R}$, *be compactly supported. Suppose that* $p \in C^k(\overline{K})$ *and* $p(0) = 0$. *Then* $pu \in H_{a-1}^k(K)$.

Lemma A.3. *(See Lemma 4.20 in [19].) The function* $r^{i\lambda_0}\Phi(\omega)\ln^s r$, *where* $\operatorname{Im}\lambda_0 = -(k-1)$ *and* $s \geqslant 0$ *is an integer, belongs to* $W^k(K \cap \{|y| < 1\})$ *if and only if it is a homogeneous polynomial in* y_1, y_2 *of degree* $k-1$.

Lemma A.4. *Let* $f \in W^k(\mathbb{R}^2)$ *and* $D^\alpha f(0) = 0$, $|\alpha| \leqslant k-2$, *if* $k \geqslant 2$. *Then there exists a sequence* $f^n \in C_0^\infty(\mathbb{R}^2)$, $n = 1, 2, \dots$, *such that* $f^n(y) = 0$ *in some neighborhood of the origin (depending on* n*) and* $f^n \to f$ *in* $W^k(\mathbb{R}^2)$.

Proof. The proof is analogous to that of Lemma 4.1 in [16]. □

A.2. Nonlocal problems in plane angles in weighted spaces

In this subsection and in the next one, we formulate some properties of solutions of problem (2.10), (2.11) in the spaces (3.6) and (3.7). First, we consider the case of weighted spaces.

For convenience, we rewrite this problem:

$$\begin{aligned} \mathbf{P}_j(y, D_y)U_j &= F_j(y) \qquad \big(y \in K_j^\varepsilon\big), \\ \mathbf{B}_{j\sigma\mu}(y, D_y)U &= \Phi_{j\sigma\mu}(y) \quad \big(y \in \gamma_{j\sigma}^\varepsilon\big). \end{aligned} \tag{A.1}$$

Along with problem (A.1), we consider the following model problem in the unbounded angles:

$$\begin{aligned} \mathcal{P}_j(D_y)U_j &= F_j(y) \qquad (y \in K_j), \\ \mathcal{B}_{j\sigma\mu}(D_y)U &= \Phi_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}). \end{aligned} \tag{A.2}$$

Lemma A.5. *(See Lemma 2.3 in [15].) Let a function* U *be a solution of problem* (A.1) (*or* (A.2)) *such that*

$$U_j \in W^{2m}\big(K_j^{D_\chi\varepsilon} \setminus \overline{\mathcal{O}_\delta(0)}\big) \quad \forall \delta > 0; \qquad U \in \mathcal{H}_{a-2m}^0\big(K^{D_\chi\varepsilon}\big),$$

where D_χ *is given by* (3.5) *and* $a \in \mathbb{R}$. *Suppose that*

$$\{F_j\} \in \mathcal{H}_a^0\big(K^\varepsilon\big), \qquad \{\Phi_{j\sigma\mu}\} \in \mathcal{H}_a^{2m-\mathbf{m}-1/2}\big(\gamma^\varepsilon\big).$$

Then $U \in \mathcal{H}_a^{2m}(K^\varepsilon)$.

Consider the asymptotics of solutions of problem (A.2).

Lemma A.6. *(See Lemma 2.1 in [13].) The function*

$$U = r^{i\lambda_0} \sum_{l=0}^{l_0} \frac{1}{l!} (i \ln r)^l \varphi^{(l_0 - l)}(\omega), \tag{A.3}$$

is a solution of homogeneous problem (A.2) *if and only if* λ_0 *is an eigenvalue of the operator* $\tilde{\mathcal{L}}(\lambda)$ *and* $\varphi^{(0)}, \dots, \varphi^{(\varkappa-1)}$ *is a Jordan chain corresponding to the eigenvalue* λ_0; *here* $l_0 \leqslant \varkappa - 1$.

Any solution of the kind (A.3) is called a *power solution*.

Lemma A.7. *(See Theorem 2.2 and Remark 2.2 in [13].) Let*

$$\{F_j\} \in \mathcal{H}_a^0(K) \cap \mathcal{H}_{a'}^0(K), \qquad \{\Phi_{j\sigma\mu}\} \in \mathcal{H}_a^{2m-\mathbf{m}-1/2}(\gamma) \cap \mathcal{H}_{a'}^{2m-\mathbf{m}-1/2}(\gamma),$$

where $a > a'$. *Suppose that the line* $\operatorname{Im}\lambda = a' - 1$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$. *If* U *is a solution of problem* (A.2) *belonging to the space* $\mathcal{H}_a^{2m}(K)$, *then*

$$U = \sum_{n=1}^{n_0} \sum_{q=1}^{J_n} \sum_{l_0=0}^{\varkappa_{qn}-1} c_n^{(l_0,q)} W_n^{(l_0,q)}(\omega, r) + U'.$$

Here $\lambda_1, \dots, \lambda_{n_0}$ *are eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$ *located in the band* $a' - 1 < \operatorname{Im}\lambda < a - 1$;

$$W_n^{(l_0,q)}(\omega, r) = r^{i\lambda_n} \sum_{l=0}^{l_0} \frac{1}{l!} (i \ln r)^l \varphi_n^{(l_0-l,q)}(\omega)$$

are the power solutions of homogeneous problem (A.2);

$$\left\{\varphi_n^{(0,q)}, \dots, \varphi_n^{(\varkappa_{qn}-1,q)}\colon q = 1, \dots, J_n\right\}$$

is a canonical system of Jordan chains of the operator $\tilde{\mathcal{L}}(\lambda)$ *corresponding to the eigenvalue* λ_n; $c_n^{(m,q)}$ *are some complex constants*; *finally,* U' *is a solution of problem* (A.2) *belonging to the space* $\mathcal{H}_{a'}^{2m}(K)$.

If the right-hand sides of problem (A.2) are of particular form, then there exist solutions of particular form. Let

$$F_j(\omega, r) = r^{i\lambda_0 - 2m} \sum_{l=0}^{M} \frac{1}{l!} (i \ln r)^l f_j^{(l)}(\omega), \qquad \Phi_{j\sigma\mu}(r) = r^{i\lambda_0 - m_{j\sigma\mu}} \sum_{l=0}^{M} \frac{1}{l!} (i \ln r)^l \varphi_{j\sigma\mu}^{(l)}, \tag{A.4}$$

where $f_j^{(l)} \in L^2(-\omega_j, \omega_j)$, $\varphi_{j\sigma\mu}^{(l)} \in \mathbb{C}$, $\lambda_0 \in \mathbb{C}$.

If λ_0 is an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, then denote by $\varkappa(\lambda_0)$ the greatest of partial multiplicities of this eigenvalue; otherwise, set $\varkappa(\lambda_0) = 0$.

Lemma A.8. *(See Lemma 4.3 in [13].) For problem* (A.2) *with right-hand side* $\{F_j, \Phi_{j\sigma\mu}\}$ *given by* (A.4)*, there exists a solution*

$$U = r^{i\lambda_0} \sum_{l=0}^{M+\varkappa(\lambda_0)} \frac{1}{l!}(i \ln r)^l u^{(l)}(\omega), \tag{A.5}$$

where $u^{(l)} \in \prod_j W^{2m}(-\omega_j, \omega_j)$*. A solution of such a form is unique if* $\varkappa(\lambda_0) = 0$ *(i.e.,* λ_0 *is not an eigenvalue of* $\tilde{\mathcal{L}}(\lambda)$*). If* $\varkappa(\lambda_0) > 0$*, then the solution* (A.5) *is defined accurate to an arbitrary linear combination of power solutions* (A.3) *corresponding to the eigenvalue* λ_0*.*

The following result is a modification of Lemma A.7 for the case in which the line $\operatorname{Im}\lambda = 1 - 2m$ contains the unique eigenvalue $\lambda_0 = i(1 - 2m)$ of $\tilde{\mathcal{L}}(\lambda)$ and this eigenvalue is proper (see Definition 3.1).

Lemma A.9. *(See Lemma 3.4 in [14].) Let* $U \in \mathcal{H}_a^{2m}(K)$*, where* $a > 0$*, be a solution of problem* (A.2) *with right-hand side* $\{F_j\} \in \mathcal{H}_a^0(K) \cap \mathcal{H}_0^0(K)$*,* $\{\Phi_{j\sigma\mu}\} \in \mathcal{H}_a^{2m-\mathbf{m}-1/2}(\gamma) \cap \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma)$*. Suppose that the band* $1 - 2m \leqslant \operatorname{Im}\lambda < a + 1 - 2m$ *contains only the eigenvalue* $\lambda_0 = i(1 - 2m)$ *of* $\tilde{\mathcal{L}}(\lambda)$ *and this eigenvalue is proper. Then* $D^\alpha U \in \mathcal{H}_0^0(K)$ *for* $|\alpha| = 2m$*.*

A.3. Nonlocal problems in plane angles in Sobolev spaces

Lemma A.10. *(See Lemma 2.4 and Corollary 2.1 in [14].) Suppose the line* $\operatorname{Im}\lambda = 1 - 2m$ *contains no eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$*. Let*

$$\{\Phi_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \quad \forall \delta > 0.$$

Then there exists a compactly supported function $V \in \mathcal{W}^{2m}(K) \cap \mathcal{H}_\delta^{2m}(K)$*, where* $\delta > 0$ *is arbitrary, such that*

$$\{\mathbf{P}_j(y, D_y)V_j\} \in \mathcal{H}_0^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma\mu}(y, D_y)V|_{\gamma_{j\sigma}^\varepsilon} - \Phi_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).$$

Now we consider the situation where the line $\operatorname{Im}\lambda = 1 - 2m$ contains the unique eigenvalue $\lambda_0 = i(1 - 2m)$ of $\tilde{\mathcal{L}}(\lambda)$ and it is proper (see Definition 3.1).

Lemma A.11. *(See Lemma 3.3 and Corollary 3.1 in [14].) Let the line* $\operatorname{Im}\lambda = 1 - 2m$ *contain only the unique eigenvalue* $\lambda_0 = i(1 - 2m)$ *of* $\tilde{\mathcal{L}}(\lambda)$ *and it is proper. Suppose that*

$$\{\Phi_{j\sigma\mu}\} \in \mathcal{W}^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \cap \mathcal{H}_\delta^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon) \quad \forall \delta > 0$$

and the functions $\Phi_{j\sigma\mu}$ *satisfy the consistency condition* (4.5)*. Then there exists a compactly supported function* $V \in \mathcal{W}^{2m}(K) \cap \mathcal{H}_\delta^{2m}(K)$*, where* $\delta > 0$ *is arbitrary, such that*

$$\{\mathbf{P}_j(y, D_y)V_j\} \in \mathcal{H}_0^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma\mu}(y, D_y)V|_{\gamma_{j\sigma}^\varepsilon} - \Phi_{j\sigma\mu}\} \in \mathcal{H}_0^{2m-\mathbf{m}-1/2}(\gamma^\varepsilon).$$

Lemma A.12. *(See Lemma 3.1 in [14].) Let the line* $\operatorname{Im}\lambda = 1 - 2m$ *contain only the proper eigenvalue* $\lambda_0 = i(1-2m)$ *of* $\tilde{\mathcal{L}}(\lambda)$*. Suppose that* $U \in \mathcal{W}^{2m}(K)$ *is a compactly supported solution of problem* (A.1) *(or* (A.2)*) and* $D^\alpha U(0) = 0$, $|\alpha| \leqslant 2m-2$*. Then the functions* $\Phi_{j\sigma\mu}$ *satisfy the consistency condition* (4.5)*.*

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